

A Unified Approach to Risk-Adjusted Performance

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1. Introduction

The most common measure of risk-adjusted performance for alpha generation strategies is the ratio of a portfolio's average active return to the standard deviation of active return, commonly known as the information ratio. The information ratio is appealing because it is easy to calculate, interpret, and apply.

Evaluating alternative strategies on the basis of the information ratio is valid so long as the probability distributions of alternative strategies differ primarily in their first two moments (mean and standard deviation). This is the case when all strategies being compared are well represented by the normal distribution. However, there can be important differences between strategies in the higher moments, namely skewness and kurtosis. For example, suppose that strategies A and B have the same information ratio, but that strategy A has one has a large amount of negative skewness and a high level of kurtosis (this is, has fat tails) and that strategy B is normally distributed. Although investors should find strategy A less desirable than strategy B, the information ratio fails to distinguish them. We need risk-adjusted performance measures that rank B superior to A.

A number of risk-adjusted performance measures have been developed to address the shortcomings of the information ratio when active return strategies are non-normal. These include the Sortino ratio, Omega, and the Stutzer index.¹ Kaplan and Knowles [2004] show that the Sortino ratio and Omega are special cases of a more general measure that they call Kappa.

The various risk-adjusted performance measures differ in theoretical motivation and mathematical form and can result in different rankings for non-normal distributions. However, they are more closely related to each other than is apparent. In this paper we unify all of these measures into a single family and expand on it. Our approach is based on expected utility theory with proportional risk aversion (PRA) with respect to active return. We say that a utility function exhibits PRA with respect to active return if the investor-specific risk aversion parameter multiplies active return in the utility function. In this way, the risk aversion parameter has the same effect on utility as the level of exposure to the alpha generator. We show that if each investor is free to set the level of exposure to the alpha generator, each risk-adjusted performance that we consider has a corresponding PRA utility function such that maximum expected utility is a monotonic transformation of the given measure.²

¹ See Sortino [2001], Shadwick and Keating [2002], and Stutzer [2000] for descriptions of the Sortino ratio, Omega, and the Stutzer index respectively.

² In the case of Omega, there is no PRA utility function that generates the measure directly, but Omega is a limiting case for a family of PRA utility functions. See section 7.

Our approach can be used to create new risk-adjusted performance measures by specifying new PRA utility functions. In particular, we present a new measure, Lambda that combines features of Kappa and the Stutzer index.

We call this group of measures the PRA family of risk-adjusted performance measures. Figure 1 presents a PRA “family tree.” An important branch of this tree is the downside risk-adjusted performance measures. Downside risk-adjusted measures only consider the possibility of returns falling below a benchmark risk. Kappa, and its special cases of the Sortino ratio and Omega are downside risk-adjusted performance measures.³ As Figure 1 indicates, our new measure, Lambda, is also a downside risk-adjusted measure, but it is not a special case of Kappa.

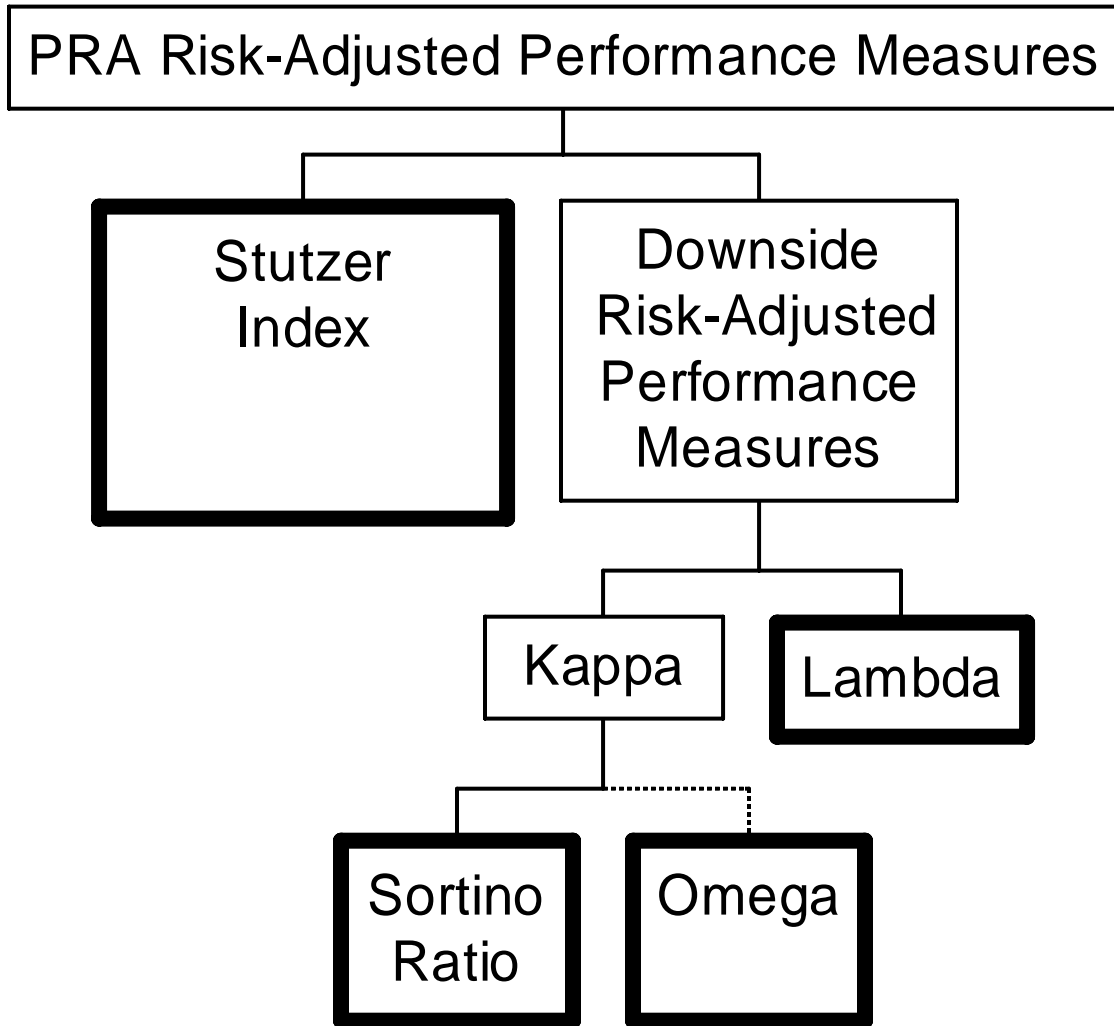
The distinctions between these risk-adjusted performance measures should only be meaningful when active returns are not normally distributed. When active returns are normally distributed, each measure is in fact a monotonic function of the information ratio. Di Pierro and Mosevich [2005] show that this is indeed the case for the Stutzer index, the Sortino ratio, Omega, and Kappa by presenting the value of each of these measures as an explicit function of the information ratio. In this paper, we present a more general result; namely that when alternative active return distributions differ only in mean and standard deviation, but not in shape (as is the case when all of the distributions are normal), all members of the PRA family of risk-adjusted performance measures are monotonic functions of the information ratio.

We illustrate the properties of the alternative risk-adjusted performance measures by estimating the information ratio, Omega, the Sortino ratio, the Stutzer index, and Lambda for a set of historical excess returns on a set of hedge funds and comparing the results graphically. We find that while the overall rankings generated by the measures are similar, each measure treats the non-normality of the excess return distributions in a distinct fashion.

The remainder of this paper is organized as follows: Section 2 sets up the framework of active return. Section 3 presents the motivation for the information ratio using mean-variance analysis. Section 4 presents the general PRA framework for creating risk-adjusted performance measures using expected utility theory. Section 5 demonstrates that all PRA measures are monotonic functions of the information ratio when distributions only differ in mean and standard deviation. Section 6 shows that the Stutzer index is a PRA risk-adjusted performance measure. Section 7 demonstrates that Kappa, with its special cases of Omega and the Sortino ratio, is also a PRA measure. Section 8 presents a general framework for downside risk-adjusted measures within the PRA framework. Section 9 presents our new downside risk-adjusted performance measure, Lambda. Section 10 shows how to estimate all of these measures using data on active returns and presents results for a set of over 300 hedge funds. Section 11 concludes the paper with a summary.

³ The dotted line connecting Kappa to Omega indicates that while Omega is a special case of Kappa as defined by Kaplan and Knowles [2004], it is limiting case of Kappa in the PRA framework.

Figure 1: Family Tree of PRA Risk-Adjusted Performance Measures



2. Active Return and Alpha Generators

Every actively managed portfolio, implicitly or explicitly, has a passive benchmark.⁴ Hence, the total return on an actively managed portfolio over a period of time (r_P) can be decomposed into the return on the passive benchmark (r_B) plus active return (a):

$$r_P = r_B + a \quad (1)$$

Active return comes from exposure to a return enhancing strategy, an alpha generator. Letting r denote the return on the alpha generator and ϕ denote the exposure to the alpha generator, we have:

$$a = \phi r \quad (2)$$

Thus active investing involves two distinct decisions: Selecting a strategy for generating active returns and deciding on the level of exposure to the chosen active strategy.

3. The Information Ratio in Mean-Variance Analysis

Expected utility theory is a standard approach in economics to model situations in which individuals need to make choices that have uncertain outcomes. The theory postulates that an individual's attitude toward taking financial risk can be described by an increasing, concave function, the utility function, which we denote as $u(\cdot)$. The individual ranks alternative decisions by calculating the mathematical expectation of $u(X)$, $E[u(X)]$, where X is the quantity of concern, the value being uncertain at the time that the decision must be made. In our application of expected utility theory, this is active return, a , as defined in equation (1).

To simplify the application of expected utility to investment decisions, Markowitz [1991(1959), 1987] developed mean-variance analysis. To apply mean-variance analysis to the problem of ranking and selecting active return strategies, we assume that the expected utility of active return for any investor is well approximated by a linear function of the expected value of a , $E[a]$, and the variance of a , $\sigma^2[a]$. Hence we assume that

$$E[u(a)] = \lambda E[a] - \frac{\lambda^2}{2} \sigma^2[a] \quad (3)$$

where λ is a parameter that reflects the investor's degree of risk aversion. We assume that λ is positive so that the investor is averse to taking risk.

⁴ The benchmark is not necessarily a single published index, but could be a portfolio of indexes, a levered or delevered position in an index or index portfolio (when the active fund's beta is not one), or some other passive portfolio. In the case of hedge funds, and other "absolute return" strategies, the benchmark is implicitly cash. See Waring and Siegel [2005] for a discussion of implicit benchmarks.

Recall that active return is the product of the portfolios' exposure to the active strategy, ϕ , and the return on the alpha generator, r . Hence,

$$E[u(a)] = \lambda\phi E[r] - \frac{\lambda^2}{2}\phi^2\sigma^2[r] \quad (4)$$

To achieve the highest possible level of expected utility using this alpha generator, the investor sets the exposure level as follows:

$$\phi = \frac{E[r]}{\lambda\sigma^2[r]} \quad (5)$$

Substituting the right-hand side of equation (5) for ϕ in equation (4) yields

$$E[u(a)] = \frac{IR^2[r]}{2} \quad (6)$$

where

$$IR[r] = \frac{E[r]}{\sigma[r]} \quad (7)$$

$IR[r]$ being the information ratio of the alpha generator, r .⁵

Note that the maximum expected utility depends only on the information ratio and not on the investor's risk aversion. Hence, under the assumptions of mean-variance analysis, all risk-averse investors, regardless of individual attitudes towards risk, should rank alpha generators using information ratios.

4. A General Framework for Risk-Adjusted Performance Measures

In this section, we create a general framework for risk-adjusted performance measures by applying expected utility theory to the problem of selecting an active return strategy. However, here we do not use approximations based on expected value and standard deviation as we did in the previous section. Rather, we specify the exact form of the utility function and maximize expected utility using the entire distribution of active returns. By using the entire distribution of active returns, we can accommodate non-normal distributions. To separate choosing an alpha generation strategy from choosing the level of exposure to it, we restrict the set of utility functions to those that exhibit proportional risk aversion (PRA) with respect to active return. Recall that a utility function exhibits PRA with respect to active return if the investor-specific risk aversion

⁵ When the benchmark is cash, $IR[r]$ is the Sharpe ratio of r .

parameter multiplies active return in the utility function. Mathematically, this means that we can write the utility function, $u(\cdot)$, in terms of an investor-specific risk-aversion parameter, λ , and an increasing concave function, $h(\cdot)$, that is the same for all investors as follows:

$$u(a) = h(\lambda a) \quad (8)$$

where a is active return. Recall that active return is the product of the return on the alpha generator, r , and the exposure to the alpha generator, ϕ . (See equation(2)). Hence,

$$E[u(a)] = E[h(\theta r)] \quad (9)$$

where

$$\theta = \lambda \phi \quad (10)$$

We can find the highest possible value of expected utility without knowledge of the risk aversion parameter by maximizing expected utility as a function of θ rather than ϕ . Let

$$\theta^*[r] = \arg \max_{\theta} E[h(\theta r)] \quad (11)$$

and

$$H[r] = E[h(\theta^*[r] r)] \quad (12)$$

$H[r]$ is the maximum expected utility that can be obtained from the alpha generator r . Note that $H[r]$ depends on the distribution of r and on the shape of the utility function, $h(\cdot)$, but not on the investor's level of risk aversion. We define $H[\cdot]$ to be our generalized risk-adjusted performance measure.

Given an investor's level of risk aversion, λ , the optimal level of exposure to r is

$$\phi = \frac{\theta^*[r]}{\lambda} \quad (13)$$

Hence by assuming that the utility function exhibits proportional risk aversion, selecting of an alpha generator and the choosing the level of exposure to the alpha generator are separate decisions in our general framework.

5. The Information Ratio in the General Framework

Although the information ratio is the most widely used risk-adjusted performance measure of active return, it is not always applicable. In our framework, it is applicable only when alternative active return distributions differ only in mean and standard deviation, but not in shape. Under these circumstances, all members of the PRA family of risk-adjusted performance measures are monotonic functions of the information ratio. To see this, let z be a random variable with mean zero and standard deviation one that has the same distribution as that which is common to all of the alpha generators being considered. Hence

$$E[h(\theta r)] = E\left[h\left(\theta\left(E[r] + \sigma[r]z\right)\right)\right] = E\left[h\left(\omega\left(1 + \frac{z}{IR[r]}\right)\right)\right] \quad (14)$$

where

$$\omega = \theta E[r] \quad (15)$$

Define

$$\omega^*(IR) = \arg \max_{\omega} E\left[h\left(\omega\left(1 + \frac{z}{IR}\right)\right)\right] \quad (16)$$

So that

$$\theta^*[r] = \frac{\omega^*(IR[r])}{E[r]} \quad (17)$$

Define

$$G(IR) = E\left[h\left(\omega^*(IR)\left(1 + \frac{z}{IR}\right)\right)\right] \quad (18)$$

So that

$$H[r] = G(IR[r]) \quad (19)$$

In Appendix A we show that $G'(IR) > 0$ if $IR > 0$. Hence, our risk-adjusted performance measure is a monotonic transformation of the information ratio when alpha generators have identically shaped distributions. An important example of this is when all of the active return distributions being considered are normal.

6. The Stutzer Index as a Special Case

Let \bar{r}_T denote the simple average of the return on an alpha generator over T periods. Stutzer [2000] notes that if $E[r] > 0$ for any single period, and the distributions of r across periods are independent and identically distributed, the probability of $\bar{r}_T \leq 0$ converges to zero as $T \rightarrow \infty$ at a computable exponential decay rate. He postulates that given a choice between such alpha generators, all investors would rank them by their probability decay rates, the higher the rate, the better. He refers to the probability decay rate of an alpha generator as its performance index. We refer to it as its ‘‘Stutzer index.’’ We denote the Stutzer index of an alpha generator r by $SI[r]$.

Stutzer shows that $SI[r]$ can be calculated as the solution to a maximization problem, namely:

$$SI[r] = \max_{\theta} -\log E[\exp(-\theta r)] \quad (20)$$

Note that the optimal value of θ in this problem is the same as $\theta^*[r]$ in equation (11) when

$$h(x) = h_{SI}(x) = 1 - \exp(-x) \quad (21)$$

Note that $h(\cdot)$ in equation (21) is an increasing concave function. In fact, using it in equation (8) results in an exponential utility with respect to active return, where λ is the constant coefficient of absolute risk aversion. From equations (12), (20), and (21), it follows that

$$SI[r] = -\log(1 - H[r]) \quad (22)$$

So although Stutzer avoids using expected utility theory to motivate $SI[r]$, it is in fact a monotonic transformation of a measure derived from expected utility.

Stutzer shows that when the alpha generator is normally distributed,

$$SI[r] = \frac{IR^2[r]}{2} \quad (23)$$

Note that this is same as maximum expected utility in the mean-variance framework. (See equation (6)).

7. Kappa as a Special Case

As an alternative to variance as a measure of risk, Harlow [1991] defines the n^{th} lower partial moment of r as⁶

$$LPM_n[r] = E\left[\max(0, -r)^n\right] \quad (24)$$

Kaplan and Knowles [2004] define the n^{th} Kappa of r as

$$K_n[r] = \frac{E[r]}{\sqrt[n]{LPM_n[r]}} \quad (25)$$

As Kaplan and Knowles show, the Sortino ratio (see Sortino [2001]) is simply the 2nd Kappa, and Omega, as defined by Shadwick and Keating [2002], is the 1st Kappa plus one.⁷

We can motivate Kappa with $n > 1$ in our general framework using the utility function defined by Fishburn [1977]. In a Fishburn utility function, the investor is risk neutral regarding performance above a given threshold and risk averse regarding performance below the threshold. With the threshold set to zero, we adopt Fishburn's utility function by setting

$$h(x) = h_n(x)x - \frac{1}{n}\max(0, -x)^n \quad (26)$$

So that

$$E[h(\theta r)] = \theta E[r] - \frac{\theta^n}{n} LPM_n[r] \quad (27)$$

If $n > 1$, $E[h(\theta r)]$ as given in equation (27) is maximized when θ is

$$\theta^*[r] = \left(\frac{E[r]}{LPM_n[r]}\right)^{\frac{1}{n-1}} \quad (28)$$

⁶ Harlow defines the lower partial moment with respect to a given threshold that we set to zero. Markowitz [1991(1959)] called $LPM_2[\cdot]$ "target semivariance" and recommended it as an alternative to variance. Later authors, such as Sortino, popularized target semivariance as "downside risk."

⁷ Independently, Darsinos and Satchell [2004] discuss a "generalized universal performance measure" that is the same as Kappa and note that it includes Omega and the Sortino ratio as special cases.

Substituting $\theta^*[r]$ as given by the right-hand side of equation (28) for θ in the right-hand side of equation (27) yields our risk-adjusted performance measure for a given value of n :

$$H_n[r] = \left(\frac{n-1}{n}\right) (K_n[r])^{\frac{n}{n-1}} \quad (29)$$

Hence, our generalized measure is a monotonic transformation of Kappa under Fishburn utility when $n > 1$.

We can motivate $K_1[r]$ (Omega minus one) as a limiting case. Let $K_n^*[r]$ denote the value of Kappa obtained by finding $H_n[r]$ through expected utility maximization and then solving equation (29) for Kappa. From equation (29) for it follows that

$$K_n^*[r] = \left(\frac{n}{n-1} H_n[r]\right)^{\frac{n-1}{n}} \quad (30)$$

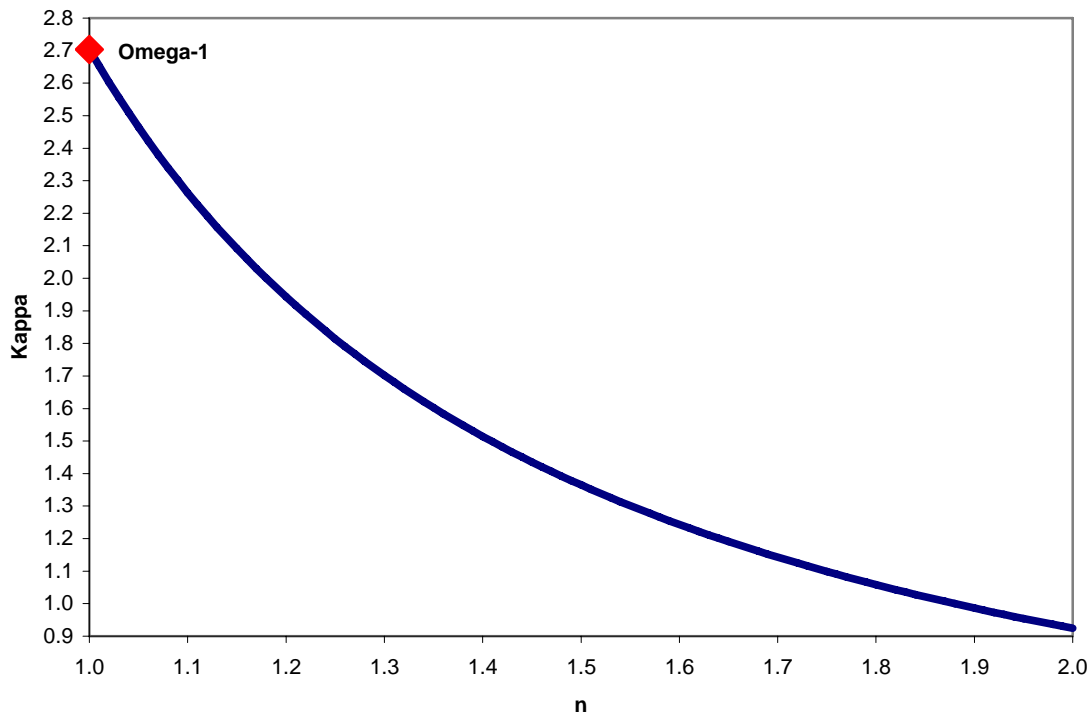
Treating n as a real number rather than as an integer, we can take the limit of $K_n^*[r]$ as $n \rightarrow 1$. We find that

$$\lim_{n \rightarrow 1} K_n^*[r] = K_1[r] \quad (31)$$

Figure 2 illustrates this convergence.

So while Omega ($K_1[r]+1$) cannot directly be derived by maximizing a PRA expected utility function, it can be derived as the limit of a family of PRA risk-adjusted performance measures.

Figure 2: Convergence of Kappa to Omega-1 as $n \rightarrow 1$



8. Generalized Downside Risk-Adjusted Performance Measures

By generalizing the Fishburn utility function, we can create a general class of downside risk-adjusted performance measures that includes the Kappa measures as special cases. We define the “generalized Fishburn utility function” as

$$h(x) = x - l(\max(-x, 0)) \quad (32)$$

where $l(\cdot)$ is a loss penalty function for when active return is below zero. To create the Fishburn utility function that we use to derive the Kappa measures shown in equation (26), we would set

$$l(y) = l_n(y) = \frac{y^n}{n} \quad (33)$$

In order for $h(\cdot)$ to be an increasing concave function, $l(\cdot)$ must be increasing and convex. Furthermore, for $h(\cdot)$ to be continuous and differentiable, we require that $l(0) = l'(0) = 0$.

The degree to which a downside risk-adjusted performance measure derived using our framework penalizes the left tail of a distribution depends on the convexity of the loss

penalty function. We define a measure of loss penalty convexity for $l(\cdot)$, $C_l(\cdot)$, as follows⁸:

$$C_l(y) = \frac{yl''(y)}{l'(y)} \quad (34)$$

Note that for $l_n(\cdot)$, $C_l(\cdot)$ is the constant $n-1$. So the higher the value of n , the greater is the penalty on the left tail in the resulting Kappa measure. Also note that we need $n>1$ in order for the loss penalty function to be convex.

The generalized Fishburn utility function allows us to decompose expected utility into an expected return component and a “downside risk” component. To do this, we take the expected value of all of the terms in equation (32) to obtain:

$$E[h(\theta r)] = \theta E[r] - E[l(\theta \max(-r, 0))] \quad (35)$$

Whenever $l(\cdot)$ can be expressed as a polynomial, the downside risk-adjusted performance measure can be expressed in terms of the Kappa measures. Since any function that is infinitely differentiable can be expressed as a polynomial with possibly an infinite number of terms by a Taylor series expansion, the relationship that we develop below is a fairly general result.

Suppose that $l(\cdot)$ can be written as a polynomial with powers from 2 to M with coefficients a_2, a_3, \dots, a_M .⁹

$$l(y) = \sum_{n=2}^M a_n l_n(y) \quad (36)$$

Recall the definition of the n^{th} lower partial moment of r , $LPM_n[r]$, given in equation (24). Using the formula for $l(\cdot)$ given by equations (33) and (36) in equation (35) and applying the definition of $LPM_n[r]$ yields the following formula for expected utility:

$$E[h(\theta r)] = \theta E[r] - \sum_{n=2}^M \frac{a_n}{n} \theta^n LPM_n[r] \quad (37)$$

From the definition of the n^{th} Kappa of r given in equation (25), we have that

$$LPM_n[r] = \left(\frac{E[r]}{K_n[r]} \right)^n \quad (38)$$

⁸ Note that this is analogous to the Arrow-Pratt measure of the relative risk aversion of a utility function.

⁹ Because $l(0)=l'(0)=0$, the lowest power is 2.

Substituting the right-hand side of equation (38) for $LPM_n[r]$ in equation (37) and substituting ω for $\theta E[r]$ (see equation (15)), yields:

$$E[h(\theta r)] = \omega - \sum_{n=2}^M \frac{a_n}{n} \left(\frac{\omega}{K_n[r]} \right)^n \quad (39)$$

To maximize expected utility, we differentiate the right-hand side of equation (39) with respect to ω and set the resulting expression equal to zero. Multiply the resulting first-order condition through by ω and rearranging terms yields:

$$\omega = \sum_{n=2}^M a_n \left(\frac{\omega}{K_n[r]} \right)^n \quad (40)$$

Substituting the right-hand side of equation (40) for the first term on the right-hand side of equation (39), and let $\omega^*[r]$ be the value of ω that solves equation (40), yields: the following formula for our risk-adjusted performance measure:

$$H[r] = \sum_{n=2}^M \left(\frac{n-1}{n} \right) a_n \left(\frac{\omega^*[r]}{K_n[r]} \right)^n \quad (41)$$

Hence, any downside risk-adjusted performance measure derived from a polynomial loss penalty function can be expressed in terms of a set of Kappas.

9. Lambda: A New Downside Risk-Adjusted Performance Measure

In the previous section, we noted that many loss penalty functions can be expressed as infinite polynomials through a Taylor series expansion. As equation (41) shows, the resulting downside risk-adjusted performance measure is a function of all integral Kappas, starting from the 2nd. One loss penalty function in particular that can be expressed as an infinite polynomial is

$$l(y) = l_\Lambda(y) = \exp(y) - y - 1 \quad (42)$$

The Taylor series expansion of $l_\Lambda(\cdot)$ can be written as

$$l_\Lambda(y) = \sum_{n=2}^{\infty} \frac{l_n(y)}{(n-1)!} \quad (43)$$

We call the downside risk-adjusted performance measure that results from this loss penalty function Lambda. We denote the Lambda of an alpha generator r as $\Lambda[r]$.

From equations (21), (32), and (42), we see that the shape of the utility function for Lambda can be expressed as

$$h(x) = h_\lambda(x) = \begin{cases} x, & \text{if } x \geq 0 \\ h_{SI}(x), & \text{if } x < 0 \end{cases} \quad (44)$$

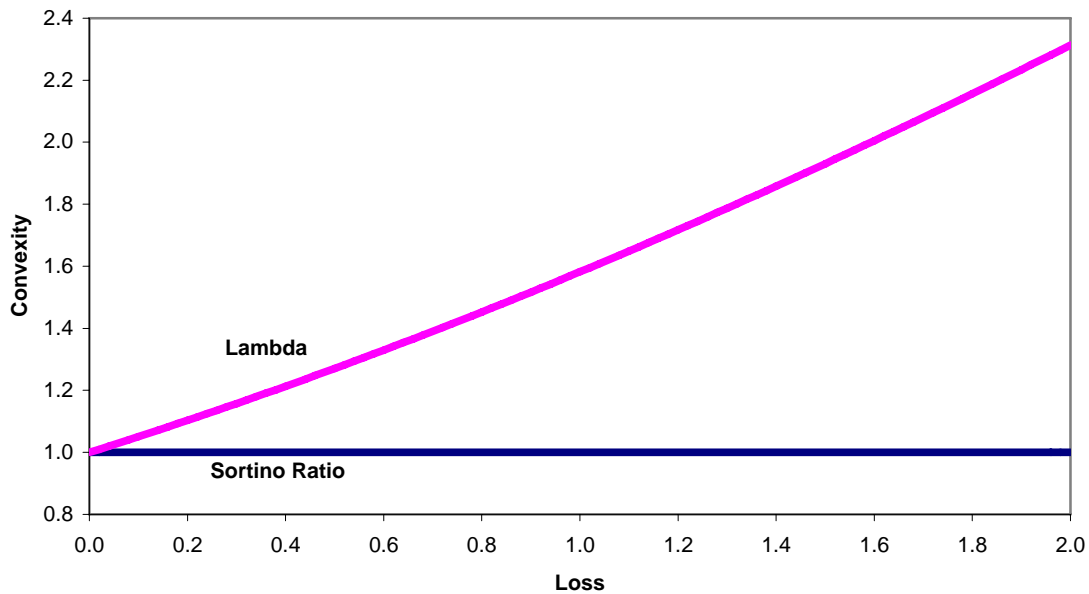
Hence, Lambda is, in effect, a downside version of the Stutzer index.

Recall that the greater the convexity of the loss penalty function, $C_l(\cdot)$, the greater the penalty the resulting downside risk-adjusted performance measure imposes on the left tail of a distribution. For the loss penalty function that generates Lambda (shown in equation (42)), the convexity is

$$C_l(y) = \frac{ye^y}{e^y - 1} \quad (45)$$

As shown in Figure 3, the convexity of the Lambda loss penalty function is greater than that of the Sortino ratio for all losses. In the next section, we show what impact this has empirically.

Figure 3: Convexity of Loss Penalty Functions of the Sortino Ratio and Lambda



10. Empirical Estimation

In this section, we estimate the information ratio, Omega, the Sortino ratio, the Stutzer index, and Lambda for a set of hedge funds, using five years of realized monthly returns in excess of the one-month riskless rate. We assume that excess returns of a given fund are independent and identically distributed across time. Letting r_t denote the realized excess return on a fund in month t , we calculate estimates of the mean (M), standard deviation (S), the 1st lower partial moment (LPM_1), and the 2nd lower partial moment (LPM_2) as follows:

$$M = \frac{\sum_{t=1}^T r_t}{T} \quad (46)$$

$$S = \sqrt{\frac{\sum_{t=1}^T (r_t - M)^2}{T - 1}} \quad (47)$$

$$LPM_1 = \frac{\sum_{t=1}^T \max(-r_t, 0)}{T} \quad (48)$$

$$LPM_2 = \frac{\sum_{t=1}^T \max(-r_t, 0)^2}{T} \quad (49)$$

where T is the number of observations (60 for our dataset).

From these summary statistics, we calculate estimates for the information ratio (IR), Omega (Ω), and the Sortino ratio (SR) as follows:

$$IR = \frac{M}{S} \quad (50)$$

$$\Omega = \frac{M}{LPM_1} + 1 \quad (51)$$

$$SR = \frac{M}{\sqrt{LPM_2}} \quad (52)$$

We estimate the Stutzer index (SI) using the technique presented by Stutzer [2000]; namely, by solving the maximization problem in equation (20), but with the empirical average replacing the mathematical expectation:

$$SI = \max_{\theta} -\log \left[\frac{\sum_{t=1}^T \exp(-\theta r_t)}{T} \right] \quad (53)$$

Finally, we estimate Lambda (Λ) by maximizing the empirical average of the utility function given in equation (32) with the loss penalty function given in equation (42). Hence,

$$\Lambda = \max_{\theta} \left[\frac{\sum_{t=1}^T \theta r_t - \left[\exp(-\max(-\theta r_t, 0)) - \max(-\theta r_t, 0) - 1 \right]}{T} \right] \quad (54)$$

Appendix B presents details on how we solve the maximization problems given in equations (53) and (54).

We obtained monthly returns on hedge funds and one-month U.S. Treasury bills for the period April 2000 through March 2005 from the Morningstar Direct institutional research platform. There were 354 funds which had returns for all 60 months. However, PRA risk-adjusted performance measures are only meaningful when the mean excess return is positive. (If it were negative, the optimal exposure in equation (11) would be negative.) Eliminating funds with negative excess returns reduces the sample to 312 funds. Furthermore, the Stutzer index and Lambda can only be calculated if there are negative excess returns in the sample. Eliminating all funds that do not have at least six negative excess returns leaves us with our sample of 307 funds.

We calculate the information ratio, Omega, the Sortino ratio, the Stutzer index, and Lambda for all 307 hedge funds in our sample using the formulas given in equations (46) through (54). In Figure 4, we plot the resulting information ratios versus Omega values. On the same graph, we plot the curve generated by formula for Omega as a function of the information ratio when active returns are normally distributed. If the sample of excess returns of a hedge fund were close to being normally distributed, its plot point would be on the curve. The non-normality of the excess returns on many hedge funds is evident in Figure 4 from the large number of points that are off the curve. Interestingly, most of the points off the curve are above it, indicating that non-normality works in favor of most of the funds when Omega is the risk-adjusted performance measure.

In Figure 5, we plot the estimated information ratios versus the estimated Sortino ratio. On the same graph, we plot the curve generated by Di Pierro and Mosevich's formula for the Sortino ratio as a function of the information ratio when active returns are normally. As in Figure 4, the non-normality of the excess returns on many hedge funds is evident in Figure 5 from the large number of points that are off the curve. However, unlike Figure 4,

the points off the curve are spread out on both sides of it. This shows that the impact of non-normality depends on the choice of risk-adjusted return measure.

In Figure 6, we plot the estimated information ratios versus the estimated Stutzer index. On the same graph, we plot the curve for the Stutzer index as a function of the information ratio when active returns are normally distributed. (See equation (23).) This figure shows that the non-normality of the excess returns is less evident with the Stutzer index than with the downside risk-adjusted measures Omega and the Sortino ratio.

In Figure 7, we plot the estimated information ratios versus the estimated values of Lambdas. On the same graph, we plot the curve for Lambda as a function of the information ratio when active returns are normally distributed. (See equation (64) in Appendix C.) The non-normality of the excess returns for many hedge funds is evident in Figure 7 from the large number of points that off the curve, largely below it. Note that as we move from a downside risk-adjusted measure with no convexity in the loss penalty function (Omega in Figure 4), to one with a constant convexity of one (the Sortino ratio in the Figure 5), to one with increasing convexity that is always greater than one (Lambda in Figure 7), the points off the curve move from being mainly above it, to being on both sides of it, to being largely below it.

Taken together, Figures 4–7 suggest that while each measure treats non-normality differently, they might result in similar rankings across the funds in the sample. The rank correlations, presented in the table below, confirm this.

Rank Correlations

Measure	<i>IR</i>	Ω	<i>SR</i>	<i>SI</i>	Λ
Information ratio	1.000				
Omega	0.993	1.000			
Sortino ratio	0.990	0.987	1.000		
Stutzer index	0.998	0.992	0.995	1.000	
Lambda	0.985	0.980	0.999	0.992	1.000

Figure 4: Information Ratio vs. Omega for Hedge Fund Sample

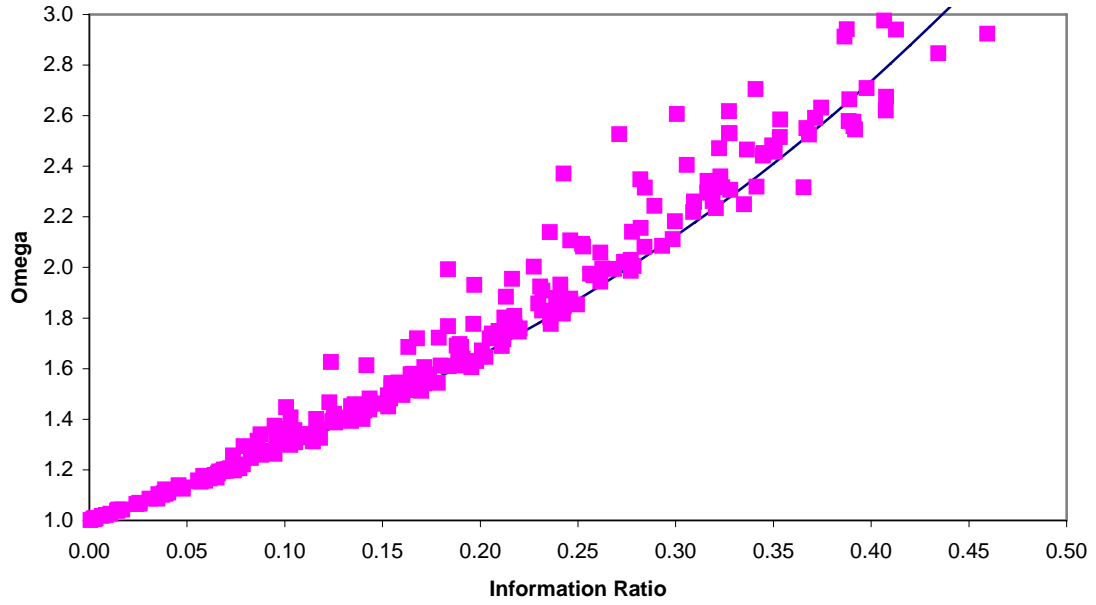


Figure 5: Information Ratio vs. Sortino Ratio for Hedge Fund Sample

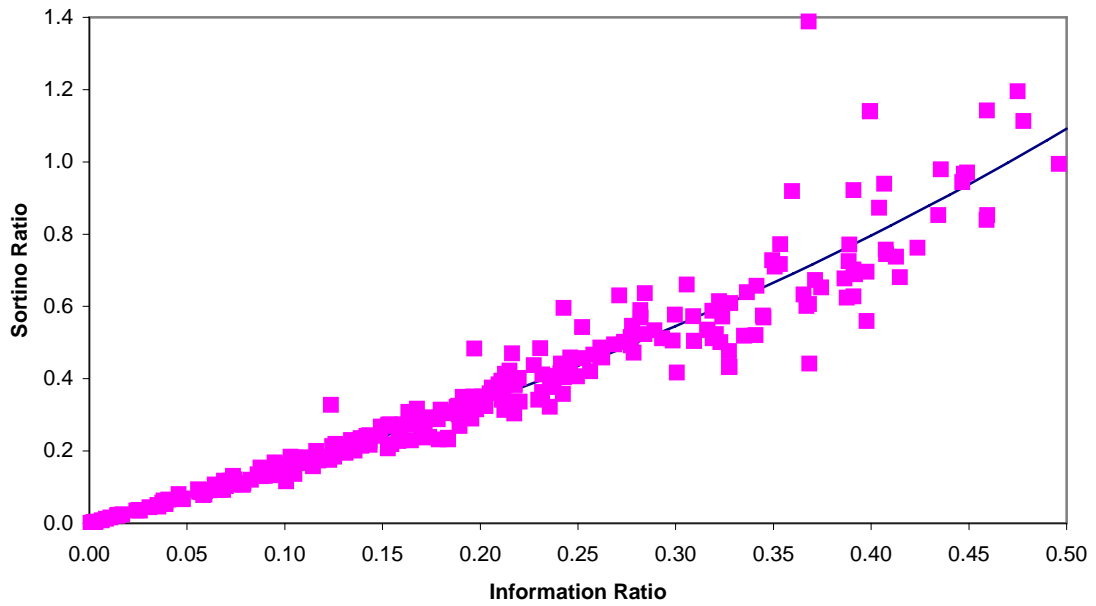


Figure 6: Information Ratio vs. Stutzer Index for Hedge Fund Sample

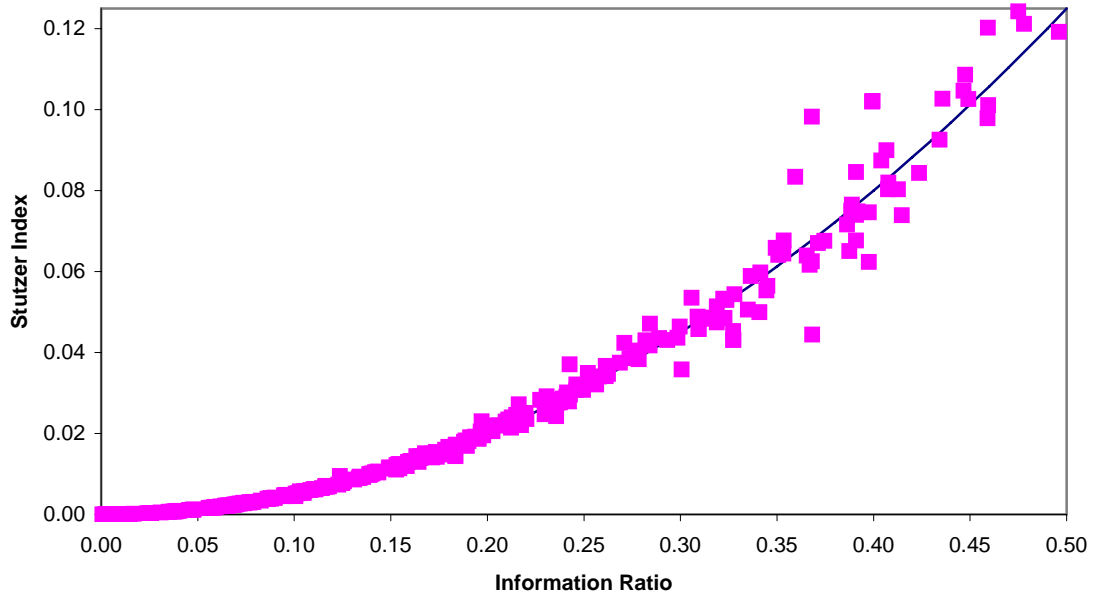
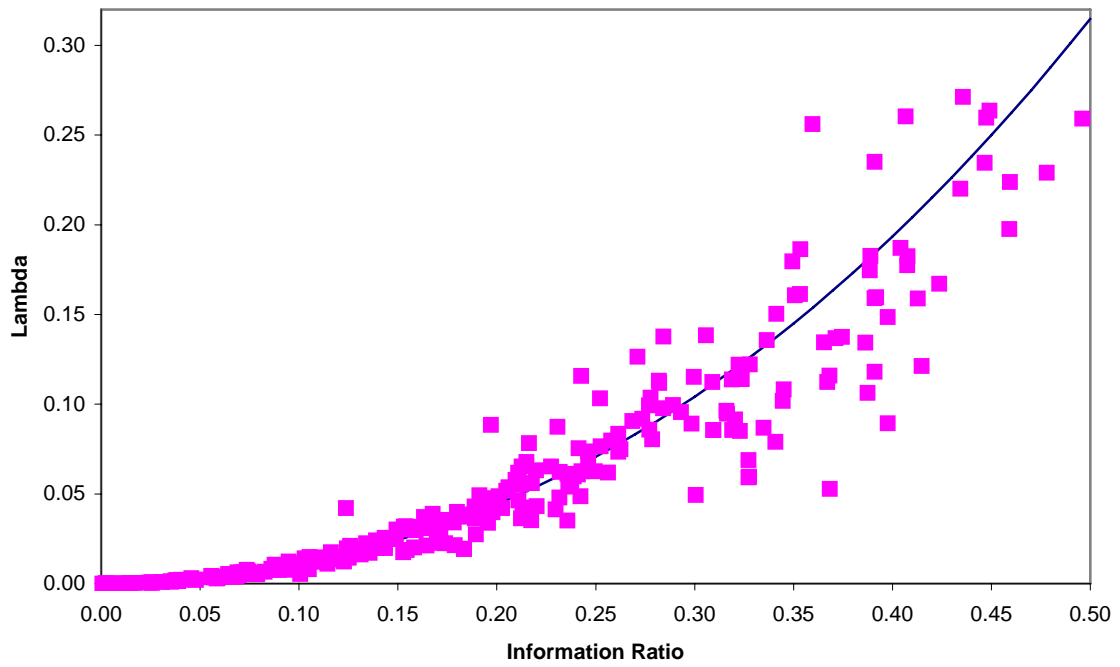


Figure 7: Information Ratio vs. Lambda for Hedge Fund Sample



11. Summary

Omega, the Sortino ratio, and the Stutzer index have all been proposed as alternatives to the information ratio for evaluating performance on a risk-adjusted basis when active return distributions are highly non-normal. While these various risk-adjusted performance measures differ in theoretical motivation and formulation, they are in fact closely related, being members of the PRA family. PRA risk-adjusted performance measures can be motivated by standard expected utility theory.

An important subset of the PRA family is the set of downside risk-adjusted performance measures. Omega and the Sortino ratio are members of a subset of downside risk-adjusted measures, the Kappa measures. Moving beyond Kappa, we define a new downside risk adjusted measure, Lambda. Lambda is in effect a downside version of the Stutzer index.

When the distributions of active returns differ only in expected value and standard deviation, but not in shape, all PRA measures are monotonic functions of the information ratio. This is the case when all active return distributions are normal. For each PRA measure, there is a function of the information ratio that yields the value of the measure when active returns are normally distributed.

Each downside risk-adjusted measure has an associated convex loss penalty function. The degree of convexity determines the amount by which the given measure penalizes the left tail of a distribution. Hence, the convexity of the loss penalty function governs the impact that non-normality has on the measure. From empirical estimates of these measures for a set of over 300 hedge funds, we find that generally the more convex the loss penalty function, the lower the measure relative to what it would be if excess returns were normally distributed. However, all of the measures rank the funds in a similar fashion, the lowest rank correlation between any pair of them being 98 percent.

Appendix A: Proof that $G'(IR) > 0$ if $IR > 0$

The maximization problem in equation (16) has the following first-order condition:

$$E \left[\left(1 + \frac{z}{IR} \right) h' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] = 0 \quad (55)$$

Differentiating both sides of equation (18) with respect to IR yields:

$$\begin{aligned} G'(IR) = & \omega^*(IR) E \left[\left(1 + \frac{z}{IR} \right) h' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] \\ & - \frac{\omega^*(IR)}{IR^2} E \left[zh' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] \end{aligned} \quad (56)$$

However, from equation (55), it follows that the first term on the right-hand side of equation (56) is zero. Therefore,

$$G'(IR) = - \frac{\omega^*(IR)}{IR^2} E \left[zh' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] \quad (57)$$

From equation (55), it follows that

$$E \left[zh' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] = -IR \cdot E \left[h' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] \quad (58)$$

From equations (57) and (58), it follows that

$$G'(IR) = \frac{\omega^*(IR)}{IR} E \left[h' \left(\omega^*(IR) \left(1 + \frac{z}{IR} \right) \right) \right] \quad (59)$$

Since $h'(\cdot) > 0$, equation (59) implies that $G'(IR) > 0$ if IR and $\omega^*(IR)$ have the same sign. Assume that $IR > 0$. In order for investors to get the benefit of an alpha generator that has a positive information ratio, they must have positive exposures to it. If an alpha generator's information ratio is positive, so is its expected value. From equations (10) and (15), it follows that $\omega^*(IR)$ is the product of a risk aversion parameter, the exposure level, and the alpha generator's expected value. Since all of these variables are positive, $\omega^*(IR)$ is positive and therefore has the same sign as IR . Therefore $G'(IR) > 0$.

Appendix B: Estimating the Stutzer Index and Lambda

To estimate a PRA risk-adjusted performance measure from a set of T observations of active returns, r_1, r_2, \dots, r_T , that we assume to be independent and identically distributed, we maximize the empirical average of expected utility:

$$H = \max_{\theta} \frac{\sum_{t=1}^T h(\theta r_t)}{T} \quad (60)$$

To solve this maximization problem, we differentiate the objective function with respect to θ and set the resulting expression to zero. This yields:

$$\sum_{t=1}^T r_t h'(\theta r_t) = 0 \quad (61)$$

Given a formula for $h'(\cdot)$, we solve equation (61) for θ numerically. We use resulting value of θ in equation (60) to compute H .

For the Stutzer index,

$$h'(x) = h'_{SI}(x) = \exp(-x) \quad (62)$$

For Lambda,

$$h'(x) = h'_{\lambda}(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ \exp(-x), & \text{if } x < 0 \end{cases} \quad (63)$$

Appendix C: Lambda under the Normal Distribution

In this appendix, we show that for normally distributed alpha generators,

$$\Lambda[r] = G_\Lambda(IR[r]) = 1 + 2 \left\{ (IR^2[r] - 1) \Phi(IR[r]) + IR[r] \Phi'(IR[r]) \right\} \quad (64)$$

where $\Phi(\cdot)$ is the cumulative standard normal cumulative density function:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left\{-\frac{t^2}{2}\right\} dt \quad (65)$$

We also show that

$$G'_\Lambda(t) = t\Phi(t) \geq 0 \quad (66)$$

Therefore, $G(\cdot)$ is an increasing function.

Let r be a normally distributed alpha generator with mean μ and standard deviation σ . From the formula for $h_\Lambda(\cdot)$ in equation (44), it follows that

$$\begin{aligned} E[h(\theta r)] &= \int_0^\infty \theta y \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\} dy \\ &\quad + \int_{-\infty}^0 (1 - \exp\{-\theta x\}) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx \end{aligned} \quad (67)$$

Changing the variable of integration in the first integral in equation (67) by setting $y = -x$ yields:

$$\begin{aligned} E[h(\theta r)] &= \int_{-\infty}^0 -\theta x \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2\right\} \\ &\quad + (1 - \exp\{-\theta x\}) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx \end{aligned} \quad (68)$$

To maximize $E[h(\theta r)]$, we differentiate the right-hand side of equation (68) with respect to θ and set the resulting expression to zero. This yields:

$$\int_{-\infty}^0 x \left[-\exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2\right\} + \exp\left\{-\left[\theta x + \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\right\} \right] dx = 0 \quad (69)$$

In order for equation (69) to hold, for all x ,

$$\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 = \theta x + \frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \quad (70)$$

Therefore,

$$\theta = \frac{2\mu}{\sigma^2} \quad (71)$$

Substituting the right-hand side of equation (71) for θ in equation (68), simplifying and rearranging terms, and defining $IR = \mu/\sigma$, we find that

$$H_{\Lambda}[r] = G_{\Lambda}(IR) = I_1(IR) + I_2(IR) - I_3(IR) \quad (72)$$

where

$$I_1(IR) = \int_{-\infty}^0 -\frac{2\mu}{\sigma^2} x \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} dx \quad (73)$$

$$I_2(IR) = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right\} dx = 1 - \Phi(IR) \quad (74)$$

$$I_3(IR) = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} dx = \Phi(IR) \quad (75)$$

To find $I_1(IR)$, we use the fact that

$$\frac{d}{dx} \left[\frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} \right] = - \left(\frac{x+\mu}{\sigma^2} \right) \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} \quad (76)$$

Rewriting equation (73) to make use of equation (76), we have

$$I_1(IR) = 2\mu \int_{-\infty}^0 \left[-\frac{x+\mu}{\sigma^2} + \frac{\mu}{\sigma^2} \right] \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} dx \quad (77)$$

Thus,

$$\begin{aligned}
I_1(IR) &= 2\mu \int_{-\infty}^0 d \left[\frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} \right] \\
&+ 2 \left(\frac{\mu}{\sigma} \right)^2 \int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{1}{2} \left(\frac{x+\mu}{\sigma} \right)^2 \right\} dx
\end{aligned} \tag{78}$$

From the fact that

$$\Phi'(z) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\}, \tag{79}$$

from the definition of IR , and from the second equality in equation (75), equation (78) simplifies as follows:

$$I_1(IR) = 2 \left[IR\Phi'(IR) + IR^2\Phi(IR) \right] \tag{80}$$

Substituting the expressions for $I_1(IR)$, $I_2(IR)$, and $I_3(IR)$, given in equations (80), (74), and (75), respectively into equation (72) and collecting like terms yields the formula for $G_\Lambda(\cdot)$ presented in equation (64). To derive the formula for $G'_\Lambda(\cdot)$ presented in equation (66), we differentiate equation (64) with respect to IR , and use the fact that

$$\Phi''(z) = -z\Phi'(z) \tag{81}$$

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