

Asset Allocation Models Using the Markowitz Approach

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Introduction

A little over forty years ago, a University of Chicago graduate student in economics, while in search of a dissertation topic, ran into a stockbroker who suggested that he study the stock market. Harry Markowitz took that advice and developed a theory that became a foundation of financial economics and revolutionized investment practice.¹ His work earned him a share of 1990 Nobel Prize in Economics.

A basic premise of economics is that, due to the scarcity of resources, all economic decisions are made in the face of trade-offs. Markowitz identified the trade-off facing the investor: risk versus expected return. The investment decision is not merely which securities to own, but how to divide the investor's wealth amongst securities. This is the problem of "Portfolio Selection;" hence the title of Markowitz's seminal article published in the March 1952 issue of the *Journal of Finance*. In that article and subsequent works, Markowitz extends the techniques of linear programming to develop the critical line algorithm.² The critical line algorithm identifies all feasible portfolios that minimize risk (as measured by variance or standard deviation) for a given level of expected return and maximize expected return for a given level of risk. When graphed in standard deviation versus expected return space, these portfolios form the efficient frontier. The efficient frontier represents the trade-off between risk and expected return faced by an investor when forming his portfolio. Most of the efficient frontier represents well diversified portfolios. This is because diversification is a powerful means of achieving risk reduction. Thus, mean-variance analysis gives precise mathematical meaning to the adage "Don't put all of your eggs in one basket."

¹ For a detailed account of the development of Markowitz's work see Peter L. Bernstein, *Capital Ideas: The Improbable Origins of Modern Wall Street*, chapter two, New York: The Free Press, 1992.

² The subsequent works include *Portfolio Selection: Efficient Diversification of Investments*, New York: John Wiley & Sons 1959 and *Mean-Variance Analysis in Choice and Capital Markets*, Oxford: Basil Blackwell Ltd, 1987. In these two books, Markowitz presents the critical line algorithm in detail.

Markowitz developed mean-variance analysis in the context of selecting a portfolio of common stocks. Over the last decade, mean-variance analysis has been increasingly applied to asset allocation. Asset allocation is the selection of a portfolio of investments where each component is an asset class rather than an individual security. In many respects, asset allocation is a more suitable application of mean-variance analysis than is stock portfolio selection. Mean-variance analysis requires not only knowledge of the expected return and standard deviation on each asset, but also the correlation of returns for each and every pair of assets. Whereas a stock portfolio selection problem might involve hundred of stocks (and hence thousands of correlations), an asset allocation problem typically involves a handful of asset classes (for example stocks, bonds, cash, real estate, and gold). Furthermore, the opportunity to reduce total portfolio risk comes from the lack of correlation across assets. Since stocks generally move together, the benefits of diversification within a stock portfolio are limited. In contrast, the correlation across asset classes is usually low and in some cases negative. Hence, mean-variance is a powerful tool in asset allocation for uncovering large risk reduction opportunities through diversification.

The relatively small data requirements of applying mean-variance analysis to asset allocation along with the speed and low cost of powerful personal computers (PCs) has led to the commercial development of many PC-based mean-variance optimization software packages for use in asset allocation. Some of these optimizers do not solve for the entire efficient frontier using the critical line algorithm; instead, they maximize a parametric objective function in mean and variance for a handful of parameter values. Other optimizers implement some form of the critical line algorithm to solve for the entire efficient frontier. The latter approach has the advantage that once the efficient frontier has been found, any number of objective functions with any number of parameter values can be optimized without having to rerun the algorithm.

The remainder of this article is in five parts as follows: Part one discusses the assumptions of mean-variance analysis. Part two illustrates how mean-variance analysis can be applied to asset allocation by using PC-based software that implements the critical line algorithm. Part three, continues the illustration by showing how some common objective functions can be easily optimized once the entire efficient frontier has been found. Finally part four concludes that while there are important pitfalls when applying mean-variance analysis, in the right hands, it is a powerful and practical application of operations research in the investment field, especially in asset allocation.

I. Assumptions of Mean-Variance Analysis

As with any model, it is important to understand the assumptions of mean-variance analysis in order to use it effectively. First of all, mean-variance analysis is based on a single period model of investment. At the beginning of the period, the investor allocates his wealth among various asset classes, assigning a non-negative weight to each asset. During the period, each asset generates a random rate of return so that at the end of the period, his wealth has been changed by the weighted average of the returns. In selecting asset weights, the investor faces a set of linear constraints, one of which is that the weights must sum to one.

Based on the game theory work of Von Neumann and Morgenstern, economic theory postulates that individuals make decisions under uncertainty by maximizing the expected value of an increasing concave utility function of consumption. In a one period model, consumption is end of period wealth. In general, maximizing expected utility of ending period wealth by choosing portfolio weights is a complicated stochastic nonlinear programming problem. Markowitz asserted that if the utility function can be approximated closely enough by a second-order Taylor expansion over a wide range of returns then expected utility will be approximately equal to a function of expected value (mean) and variance of

returns.³ This allows the investor's problem to be restated as a mean-variance optimization problem so that the objective function is a quadratic function of portfolio weights.

The utility function is assumed to be increasing and concave because we assume that (1) investors prefer more consumption to less, and (2) investors are risk averse. In terms of the approximating utility function, this translates into expected utility being increasing in expected return (more is better than less) and decreasing in variance (the less risk the better). Hence, of all feasible portfolios, the investor should only consider those that maximize expected return for a given level of variance, or minimize variance for a given level of expected return. These portfolios form the mean-variance efficient set.

II. Illustration of Mean-Variance Analysis

To illustrate mean-variance analysis as it applies to asset allocation, consider an investor whose portfolio is entirely in the U.S. capital markets but is considering going into non-U.S. markets. The first step of the analysis is to divide the world capital markets into broad asset classes. In this example, we have four: U.S. stocks, non-U.S. stocks, U.S. bonds, and non-U.S. bonds. The second step is to develop capital market assumptions; namely, expected returns and standard deviations for each asset class and correlations between each pair of asset classes. These assumptions are usually derived from historical data on asset class returns and current market conditions. The third step is to generate the efficient frontier by running the critical line algorithm.

Markowitz showed that while there are infinitely many efficient portfolios, you only need a limited number of corner portfolios to identify all efficient portfolios. Not all efficient portfolios contain all assets.

Moving along the efficient frontier, a corner portfolio is located where an asset weight or slack variable is

³ See *Mean-Variance Analysis*, page 60.

either added or dropped. Every efficient portfolio is a linear combination of the two corner portfolios immediately adjacent to it. Thus, by locating all corner portfolios, the critical line algorithm generates the entire efficient frontier.

Figure 1 depicts the efficient frontier for our example of an asset allocation problem. The first four corner portfolios are labeled CP1, CP2, CP3, and CP4 respectively. The minimum variance portfolio is labeled MVP. The first corner portfolio, CP1, is 100% in the asset class with the highest expected return and highest level of risk, non-U.S. stocks. In order to obtain portfolios with lower risk and lower expected return, another asset class must be added, U.S. stocks. All efficient portfolios between CP1 and CP2 are entirely made up of the two classes of stocks. At CP2 which is 25.0% U.S. stocks and 75.0% non-U.S. stocks, a third asset class is added, non-U.S. bonds. Non-U.S. bonds displace the two classes of stocks between CP2 and CP3, so that at CP3, the portfolio consists of 39.1% U.S. stocks, 14.3% non-U.S. stocks, and 46.6% non-U.S. bonds. At CP3, U.S. bonds are added so that between CP3 and CP4, all four asset classes are represented. At CP4, which consists of 4.3% U.S. stocks, 65.7% U.S. bonds, and 30.0% non-U.S. bonds, non-U.S. stocks are dropped. At the fifth corner portfolio, which consists of 72.6% U.S. bonds and 27.4% non-U.S. bonds, U.S. stocks are dropped. (This portfolio is not shown due to its proximity to MVP). The minimum variance portfolio, MVP, occurs shortly thereafter and consists of 73.3% U.S. bonds and 26.7% non-U.S. bonds.

In addition to the efficient frontier, Figure 1 also shows the available asset classes and the investor's U.S.-only portfolio (labeled "Current"), 60% U.S. stocks and 40% U.S. bonds.

The graph makes it apparent that investing in only one asset class or a subset of asset classes is highly inefficient. The low correlations between asset classes (see Table 1) make diversification a wise investment strategy.

The efficient portfolios in this example contain large proportions of non-U.S. stocks. In reality, U.S. investors typically face limits on how much they can invest in non-U.S. assets. These limits can be incorporated into the analysis by adding linear constraints to the optimization problem. The result would

be an efficient frontier with inferior opportunities to the one shown in Figure 1, but in which all efficient portfolios satisfy the investor's constraints.

III. Objective Functions

Once the efficient frontier has been generated, one efficient portfolio must be chosen. The type of full fledged utility analysis discussed in part one is not usually used in practice. Instead, simpler mean-variance objective functions are used. Many of these functions can be readily optimized by using a list of the corner portfolios. Some of these functions are described below.

Maximize Expected Return for a Given Level of Risk

By definition, every efficient portfolio maximizes expected return for a given level of risk. By inspecting the list of corner portfolios, we can find the pair of adjacent corner portfolios that describe the section of the efficient frontier that contains the portfolio with a given level of risk. The desired portfolio itself is a linear combination of the two corner portfolios. The weights can be found by solving a quadratic equation.

Suppose our investor wanted to hold the efficient portfolio that has the same level of risk as his current portfolio of 60% U.S. stocks and 40% U.S. bonds. From Figure 1, we can see that his current portfolio has a standard deviation of 14.5% and an expected return of 12.4%. Moving upward to the efficient frontier, he could hold a more diversified portfolio with the same risk but an expected return of 14.7%.

Risk Tolerance

A common technique for selecting a portfolio is to maximize a linear function of expected return and variance, namely

$$\text{Expected Return} - \frac{\text{Variance}}{\text{Risk Tolerance}}$$

Risk tolerance is a nonnegative number that measures how the investor views the trade-off between risk (as measured by variance) and expected return. The point labeled “Max EU” in Figure 1 is optimal for an investor with a risk tolerance of 50%.

Safety-First

In 1952, Andrew D. Roy suggested that an investor select a portfolio so as to maximize the probability of achieving at least a given return⁴. The idea is that if return falls below the threshold there will be some dire consequence. Hence, this model of investor behavior is called safety-first.

Drawing the efficient frontier in standard deviation-expected return space, the portfolio which maximizes the probability of realized return being greater than the threshold can be found by identifying the straight line passing through the expected return axis at the threshold that is tangent to efficient frontier. The portfolio at the point of tangency is the desired portfolio. In Figure 1, the point labeled “Max P[R>5]” represents the portfolio that maximizes the probability of achieving a return of at least 5%.

IV. Caveats and Conclusion

Mean-variance analysis provides a powerful framework for asset allocation. However, like all powerful analytical techniques, there are certain caveats which must be considered when applying it.

The Markowitz model treats expected returns, standard deviations, and correlations as population parameters. However, in practice population parameters are unavailable. Instead, statistical estimates must be used. The estimation errors introduced can distort the optimization results. It can be shown that even small estimation errors can result in large deviations from optimal allocations in an optimizer's results.⁵

⁴ Andrew D. Roy, “Safety First and the Holding of Assets,” *Econometrica*, 20, pp. 431-49.

⁵ See Richard O. Michaud, “The Markowitz Optimization Enigma: Is ‘Optimized’ Optimal?” *Financial Analysts Journal*, January/February, 1989; and Scott L. Lummer, Mark W. Riepe, and Laurence B. Siegel, “Taming Your Optimizer: A Guide Through the Pitfalls of Mean-Variance Optimization,” in *Global Asset Allocation: Techniques for Optimizing Portfolio Management*, edited by Jess Lederman and Robert A. Klein, John Wiley & Sons, 1994.

Being aware of the sensitivity of the results of mean-variance optimization to statistically formed inputs is an important aspect of applying the technique. Historical statistics should not be blindly fed into an optimizer. Inputs should be adjusted to reflect a sound understanding of capital markets.

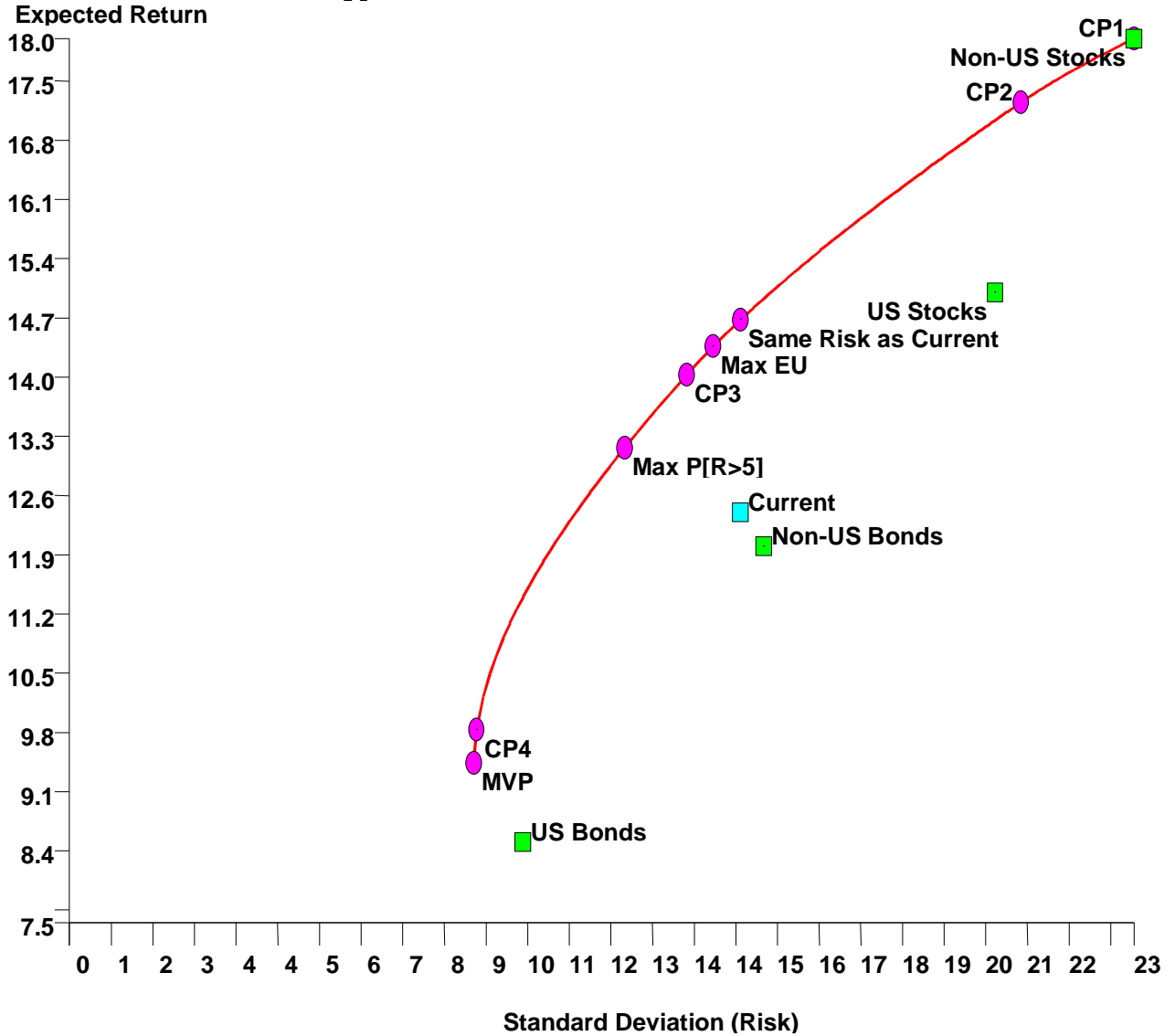
Another caveat in applying mean-variance analysis is that the results can sometimes be unstable. A small move along the efficient frontier could correspond to a large change in asset allocation (which generally should not be implemented). This kind of instability happens when the correlation matrix is ill-conditioned. The most common cause of the correlation matrix being ill-conditioned is the inclusion of very similar assets in the analysis.

Mean-variance analysis has gained wide spread acceptance among practitioners of asset allocation. This is largely due to the fact that the technology is readily available in the form of PC software. Unfortunately, there are certain caveats that make it easily subject to misuse. However, when used with a knowledge of capital market behavior and wisdom in choice of asset classes, its power can be harnessed to great practical advantage.

Table 1: Capital Market Assumptions

Asset Class	Expected Return	Standard Deviation	Correlation with			
			U.S. Stocks	Non-U.S. Stocks	U.S. Bonds	Non-U.S. Bonds
U.S. Stocks	15.0	20.0	1.00			
Non-U.S. Stocks	18.0	23.0	0.58	1.00		
U.S. Bonds	8.5	9.8	0.54	0.21	1.00	
Non-U.S. Bonds	12.0	15.0	0.03	0.52	0.15	1.00

Figure 1: Efficient Frontier



Source: Generated by the author using Ibbotson Associates' *EnCorr Optimizer* software with the capital market assumptions in Table1.