Equity Performance Attribution Methodology

Morningstar Methodology Paper

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Contents

Version History

May 2011	Updated for the addition of transaction-based attribution and contribution.
March 2009	Updated for the addition of the three-factor model.
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Introduction

Overview

Performance attribution analysis consists of comparing a portfolio's performance with that of a benchmark and decomposing the excess return into pieces to explain the impact of various portfolio management decisions. This excess return is the active return. For a portfolio denominated in the investor's home currency, the investment manager's active return is decomposed into weighting effect, selection effect, interaction, transaction effect, and residual. Weighting effect refers to the portion of an investment manager's value-add attributable to the manager's decision on how much to allocate to each market sector, or in other words, a manager's decision to overweight and underweight certain sectors compared with the benchmark. The selection effect represents the portion of performance attributable to the manager's stock-picking skill. Interaction, as its name suggests, is the interaction between the weighting and the selection effects and does not represent an explicit decision of the investment manager. The transaction effect captures the portion of performance attributable to trade execution and can only be calculated if transaction information is provided. A holding-based attribution analysis is performed when transaction information is not provided, and this type of analysis may produce residuals. Residual is the portion of the return that cannot be explained by the holdings composition at the beginning of the analysis period, and this gap is usually caused by intraperiod portfolio transactions, security corporate actions, and so on. Attribution analysis focuses primarily on the explicable part of the active return--the weighting, selection, and interaction. Appendix A is dedicated to contribution, a topic that is often associated with attribution. The transaction effect is presented in Appendix B, and residual is discussed in Appendix C.

Introduction (Continued)

This document first reviews the classic attribution approaches of Brinson, Hood, and Beebower and Brinson and Fachler, the principles upon which today's performance attribution methodologies are founded. The next sections present three attribution approaches: top-down, bottom-up, and three-factor. In addition, each of these three approaches can be implemented using the arithmetic or the geometric method. These six combinations and their uses are described in the subsequent sections, followed by how these attribution results can be accumulated in a multiperiod analysis. Although multiple alternatives are presented in this document, the recommended method of Morningstar is the top-down geometric method. The top-down approach presents a uniform framework for comparing multiple investment managers, and the geometric method has the merit of theoretical and mathematical soundness. This document focuses on equity attribution performed in the portfolio's base currency, and topics such as fixed income and currency attribution analyses are outside of the scope of this document.

Effects Versus Components

When performing attribution analysis, it is important to distinguish between effects and components. An effect measures the impact of a particular investment decision. An effect can be broken down into several components that provide insight on each piece of an overall decision, but each piece in isolation cannot represent the investment manager's decision. For example, an investment manager may make an active decision on sector weighting by overweighting certain sectors and underweighting other sectors. As overweighting certain sectors necessitates underweighting others and vice versa, the decision is on the entire set of sector weightings. To better understand the sector-weighting effect, one may examine the impact of individual sectors as components that provide additional insight. However, each of these components cannot be used in isolation to measure the impact of a decision, as it is not meaningful to say that an investment manager made a particular decision to time exposure to the cyclical sector, for example.

Review of the Classic Approach--Brinson, Hood, and Beebower
Today's approaches to performance attribution are founded on the principles presented in an article[1] written by Brinson, Hood, and Beebower (BHB) and published in 1986. Therefore, it is important to review the BHB model even though the model in its original form is not adopted. The study is based on the concept that a portfolio's return consists of the combination of group (for example, asset class) weightings and returns, and decision-making is observed when the weightings or returns of the portfolio vary from those of the benchmark. Thus, notional portfolios can be built by combining active or passive group weightings and returns to illustrate the value-add from each decision.

The study deconstructs the value-added return of the portfolio into three parts: tactical asset allocation, stock selection, and interaction. The formulas for these terms are defined below:

Tactical Asset Allocation	=	II - I	=	$\sum (w_i^P - w_i^B) \bullet R_i^B$
Stock Selection	=	III - I	=	$\sum w_i^B \bullet (R_i^P - R_i^B)$
Interaction	=	IV - III - II + I	=	$\sum (w_i^P - w_i^B) \bullet (R_i^P - R_i^B)$
Total Value Added	=	IV - I	=	$\sum w_i^P \bullet R_i^P - w_i^B \bullet R_i^B$

These formulas are based on four notional portfolios. These notional portfolios are constructed by combining different weightings and returns, and they are illustrated in the chart below:

Weights \ Returns	Portfolio	Benchmark
Portfolio	IV $\sum \boldsymbol{w_j^P \bullet R_j^P}$ Portfolio Return	II $\sum \boldsymbol{w_j^P \bullet R_j^B}$
Benchmark	III $\sum \boldsymbol{w_j^B \bullet R_j^P}$	I $\sum \boldsymbol{w_j^B \bullet R_j^B}$ Benchmark Return

[1] Brinson, Gary P., L. Randolph Hood, and Gilbert L. Beebower, "Determinants of Portfolio Performance," *Financial Analysts Journal*, July-August 1986, pp. 39-44.

Introduction (Continued)

Where:

w_j^B	=	The benchmark's weighting for group j
w_j^P	=	The portfolio's weighting for group j
R_j^B	=	The benchmark's return for group j
R_j^P	=	The portfolio's return for group j

The tactical asset-allocation effect, also known as the weighting effect, is the difference in returns between notional portfolios II and I. Notional portfolio II represents a hypothetical tactical asset allocator that focuses on how much to allocate to each group but purchases index products for lack of opinions on which stocks would perform better than others. Notional portfolio I is the benchmark, which, by definition, has passive group weightings and returns. These two notional portfolios share the same passive group returns but have different weightings; thus, the concept intuitively defines the weighting effect as the result of active weighting decisions and passive stock-selection decisions.

The stock-selection effect, also known as the selection effect, is the difference in returns between notional portfolios III and I. Notional portfolio III represents a hypothetical security-picker that focuses on picking the right securities within each group but mimics how much money the benchmark allocates to each group because the person is agnostic on which groups would perform better. As described above, notional portfolio I is the benchmark, which has passive group weightings and returns. These two notional portfolios share the same passive group weightings but have different group returns; thus, the concept intuitively defines the selection effect as the result of passive weighting decisions and active stock-selection decisions.

While the weighting and selection effects are intuitive, the interaction portion is not easily understood. The interaction term, as its name suggests, is the interaction between the weighting and the selection effects and does not represent an explicit decision of the investment manager. Due to its apparent lack of meaning, Morningstar believes that it is a better practice to incorporate it into either the weighting or the selection effect, whichever of the two represents the secondary decision of the investment manager. The concept of primary versus secondary decision is discussed in more details in the next section.

The Morningstar methodology for equity performance attribution is founded on the principles of the BHB study, but the BHB model in its original form is not adopted. First, the BHB model is an asset-class-level model and does not break down attribution effects into group-level components. The next section presents the Brinson and Fachler model; this method addresses group-level components. Furthermore, much has evolved in the field of performance attribution since the BHB study. Methodologies are needed to incorporate the interaction term into the other two effects, accommodate for multiple hierarchical weighting decisions, perform multiperiod analysis, and so on. These topics are addressed in subsequent sections of this document.

Review of Attribution Components--Brinson and Fachler

The BHB model presented in the previous section shows how attribution effects are calculated. As discussed in the "Effects versus Components" section of this document, an effect can be broken down into several components. Today's approaches to component-level attribution are based on concepts presented in a study[2] by Brinson and Fachler (BF) in 1985. In this article, the impact of a weighting decision for a particular group j is defined as

$$(w_j^P - w_j^B) \bullet (R_j^B - R^B),$$

where $(w_j^P - w_j^B)$ is the same as the equation for the tactical asset-allocation effect in the BHB study. It is the difference between the portfolio's weighting in this particular group and the

[2] Brinson, Gary P., and Nimrod Fachler, "Measuring Non-US Equity Portfolio Performance," *Journal of Portfolio Management*, Spring 1985, pp.73-76.

benchmark's weighting in the same group, representing the investment manager's weighting decision. In the BHB model, $(w_j^P - w_j^B)$ is multiplied by the benchmark's total return. This basic principle is preserved in the BF model as the latter also uses the benchmark return. However, in order to gain insight into each group's value-add, the term is transformed into the return differential between the group in question and the total return. Thus, this term intuitively illustrates that a group is good if it outperforms the total. This formula is not in conflict with the BHB model because their results match at the portfolio level; in other words, the sum of BF results from all groups equals the BHB tactical asset-allocation effect.

With the two multiplicative terms of the formula combined, the BF formula illustrates that it is good to overweight a group that has outperformed and underweight a group that has underperformed. This is because overweighting produces a positive number in the first term of the formula, and outperformance yields a positive number in the second term, leading to a positive attribution result. Similarly, a negative weighting differential of an underweighting combined with a negative return differential of an underperformance produces a positive attribution result. Furthermore, it is bad to overweight a group that has underperformed and underweight a group that has outperformed because these combinations produce negative results. This concept is illustrated in the chart below

	Underperform	Outperform
Overweight	−	+
Underweight	+	−

Top-Down Versus Bottom-Up Approach

There are several approaches to performance attribution, and we focus on two of them: top-down and bottom-up. These are two-factor models, decomposing active return into weighting effect and selection effect. In addition, we present the three-factor model in this document as an alternative to these two approaches. In a three-factor model, the active return is deconstructed into three components, displaying the BHB Interaction term as the third factor. The choice between the top-down and the bottom-up approaches depends on the investment decision process of the portfolio being analyzed, and the three-factor model takes an agnostic view on the order of the investment decision process.

The top-down approach to portfolio attribution is most appropriately used when analyzing an investment manager with a top-down investment process that focuses on one or multiple weighting allocation decisions prior to security selection. In this decision-making process, the weighting effect is primary and the selection effect is secondary. As discussed in the BHB section, the interaction term of the BHB approach is incorporated in the effect of the secondary decision, which is the selection effect in this case.

The bottom-up approach is most relevant in analyzing an investment manager with a bottom-up process that emphasizes security selection. In this decision-making process, the selection effect is primary, and the weighting effect is secondary. Unlike the top-down approach, which can measure the effects of multiple weighting allocation decisions, there is only the weighting effect in the bottom-up approach. As discussed in the BHB section above, the interaction term of the BHB approach is included in the effect of the secondary decision, which is the weighting effect in this case.

Both the top-down and bottom-up approaches involve hierarchical decision. For example, in the case of a top-down analysis, an investment manager may first decide on regional weighting, followed by sector weighting and market-capitalization weighting, before making security selections. The analysis is hierarchical because the weighting at each decision level is anchored upon the weighting of the prior decision. Similarly, in a bottom-up analysis, an investment manager first decides on security selection before making weighting decisions such as sector weighting.

Introduction (Continued)

Arithmetic Versus Geometric Attribution

As attribution effects are the results of the portfolio's relative weighting and performance to those of the benchmark, the return comparison can be performed using an arithmetic or geometric method. The arithmetic method refers to simple subtractions of return terms in formulas and is very intuitive; however, it works best in a single-period analysis, and additional "smoothing" is required to apply it in a multiperiod setting. Refer to the "Multiple Period Analysis" section of this document for details. The geometric method takes a geometric difference by translating returns into "return relatives" (that is, one plus the return), performing a division of the two return relatives, and subtracting 1 from the result. It is more complicated than the arithmetic method but has the benefit of being theoretically sound for both single-period and multiperiod analyses when applied to effect statistics.

Example

As the top-down approach is more complex, as it may involve a hierarchy of weighting decisions, it is more helpful to provide an example that illustrates this process throughout the document. Let us assume a simple example in which the investment process consists of decision-making in the following order:

Decision Level	Decision	Choice
1	Regional weighting	Asia versus Europe
2	Sector weighting	Cyclical versus defensive
3	Market-cap weighting	Large cap versus small cap
4	Security selection	

Note

- ► All formulas assume that the individual constituents and the results are expressed in decimal format. For example, the number 0.15 represents 15%.

Basic Mathematical Expressions

Formulas

As attribution formulas use many mathematical expressions in common, these mathematical expressions and their formulas are defined in this section and are used throughout the document.

[1] $$w_g^B = \begin{cases} \dfrac{\text{benchmark wt of stock g}}{w_{\text{Ø}}^B} & \text{if } |\text{g}| = M \\ \displaystyle\sum_{h \in \Omega_g} \frac{w_h^B}{w_{\text{Ø}}^B} & \text{if } |\text{g}| < M \end{cases}$$

[2] $$w_g^P = \begin{cases} \dfrac{\text{portfolio wt of stock g}}{w_{\text{Ø}}^P} & \text{if } |\text{g}| = M \\ \displaystyle\sum_{h \in \Omega_g} \frac{w_h^P}{w_{\text{Ø}}^P} & \text{if } |\text{g}| < M \end{cases}$$

[3] $$R_g^B = \begin{cases} \text{return on stock g} & \text{if } |g| = M \\ \dfrac{\displaystyle\sum_{h \in \Omega_g} w_h^B \bullet R_h^B}{w_g^B} & \text{if } |g| < M \end{cases}$$

[4] $$R_g^P = \begin{cases} \text{return on stock g} & \text{if } |g| = M \\ \dfrac{\displaystyle\sum_{h \in \Omega_g} w_h^P \bullet R_h^P}{w_g^P} & \text{if } |g| < M \end{cases}$$

Basic Mathematical Expressions (Continued)

Where:

w_g^B	=	The benchmark's weighting for group g
w_g^P	=	The portfolio's weighting for group g
R_g^B	=	The benchmark's return for group g
R_g^P	=	The portfolio's return for group g
g	=	The vector that denotes the group
$\lvert g\rvert$	=	The number of elements in the vector g, representing the hierarchy level of the group
M	=	The level that represents the security level, that is, the last grouping hierarchy
Ω_g	=	All of the subgroups within the group g that are one hierarchy level below
$\varnothing$	=	The total level, which is the equity portion of the portfolio or benchmark

Explanation of Formulas

A group represents a basket of securities classified by the end user, such as economic sector, market cap, P/E, region, country, and so on. "Group" is the most generic term that represents a group of securities or a single security. The g symbol represents the vector that denotes the group. In our example, Europe's cyclical sector's small cap is denoted as (2,1,2) because it is the second region's first sector's second market-cap bucket. This particular market cap's fifth stock is denoted as (2,1,2,5). When $g = \varnothing$, it represents the null set that denotes the total level such as the total equity portfolio or the total equity benchmark.

The variable $\lvert g\rvert$ is the number of elements in the vector g, representing the decision level of the group in the hierarchy. In our example, $\lvert g\rvert = 2$ stands for the sector level because it is the second level of decision. Note that $\lvert g\rvert = 0$ represents the total level such as the total equity benchmark or the total equity portfolio. The variable M denotes the level that represents the security level, that is, the last decision in the hierarchy. In our example, $M = 4$ because security level is the fourth level of decision.

The expression Ω_g represents all of the subgroups within the group g that are one hierarchy level below. Think of a family tree and let each decision level be a generation of relatives; the Ω_g symbol represents all of the children of the same parent g. In our example, Asia is denoted by $g = (1)$. When using formula [1] to calculate the benchmark weighting of Asia, $\Omega_{(1)}$ represents all of the subgroups within Asia. They are cyclical and defensive sectors, denoted by $g = (1,1)$ and $g = (1,2)$, correspondingly. The second part of formulas [1] and [2] simply states that the weighting of Asia is the sum of the weightings of the Asian cyclical and Asian defensive sectors. Similarly, the second part of formulas [3] and [4] means that the return of Asia is the weighted sum of the returns of Asian cyclical and Asian defensive.

Special Situation I: Groups Without Holdings

- If neither the portfolio nor the benchmark has holdings in a particular group, this group should be ignored in order to provide a meaningful attribution analysis.

- If the portfolio does not have holdings in a particular group but the benchmark does, the group's portfolio weighting is zero, and the group's portfolio return is assumed to be the same as the group's benchmark return. This rule applies regardless of whether the group represents long or short positions. For example, if the sector-weighting decision is being evaluated, and the portfolio does not have holdings in the Asian cyclical sector while the benchmark does, the portfolio's return in the Asian cyclical sector is assumed to be the same as the benchmark's return in the Asian cyclical sector. The active return is attributable entirely to the sector weighting effect and not subsequent decisions such as market-cap-weighting and security selection in a top-down model. Similarly, in a bottom-up or three-factor model, the active return is attributable entirely to the sector-weighting effect and not to the security-selection effect. This makes intuitive sense as the decision to differ from the benchmark's weighting is a weighting effect.

- If the portfolio has holdings in a particular group but the benchmark does not have holdings in the same group, the group's benchmark return is assumed to be the same as the group's portfolio return. The only exception to this rule is the short position situation described below.

Special Situation II: Short Positions

- When the portfolio or the benchmark has short positions, attribution analysis must be performed on the short positions separately from the long positions. In other words, short positions and long positions are in separate groups, and the number of groups is potentially double that of an analysis where only long positions are present. To ensure that the separation is clear, long and short positions must be separated at the first level of the decision hierarchy. For example, when the first level of the decision hierarchy is regional allocation, and the regional classifications are Asia and Europe, a portfolio containing short positions should have four regional classifications: Asia long, Europe long, Asia short, and Europe short.

- For levels of the decision hierarchy other than the security level ($|g| < M$), when the benchmark does not have holdings in a particular short position group, this group's benchmark's return is assumed to be the same as the benchmark's return of the same group's long position counterpart in order to allocate effects correctly. For example, if the benchmark does not have short position holdings in the Asian cyclical sector, the return of this sector is assumed to be the same as the benchmark's return in the long positions of the Asian cyclical sector.

Top-Down Approach for Single Period

Overview
As discussed in the Introduction, the top-down approach to portfolio attribution is most appropriately used when analyzing an investment manager with a top-down investment process that focuses on one or multiple weighting allocation decisions prior to security selection. These decisions are hierarchical. In our example, the investment manager first decides on regional weighting, followed by sector weighting and market-cap weighting before making security selections. In this decision-making process, the weighting effect is primary, and the selection effect is secondary.

This section addresses the top-down approach in a single-period attribution analysis. The single-period methodology serves as a foundation for the multiperiod attribution, and the latter is discussed in the last section of this document.

Attribution can be performed using the arithmetic or geometric method. These methods and their merits are discussed in the Introduction. This section focuses on the presentation and the explanation of the formulas.

Top-Down Approach for Single Period (Continued)

Arithmetic Method

[5] $$CA_g = (w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} \bullet w_g^B) \bullet (R_g^B - R_{\bar{g}}^B)$$

[6] $$EA_{g,n} = \begin{cases} \sum_{h \in \Omega_g} CA_h & \text{if } n = |g| + 1 \\ \sum_{h \in \Omega_g} EA_{h,n} & \text{if } n > |g| + 1 \end{cases}$$

[7] $$AA_\text{Ø} = R_\text{Ø}^P - R_\text{Ø}^B = \sum_{n=1}^{M} EA_{\text{Ø},n}$$

Where:

CA_g	=	Component attributable to group g, calculated based on arithmetic method
$EA_{g,n}$	=	Effect attributable to group g at decision level n, based on arithmetic method
$AA_\text{Ø}$	=	The portfolio's active return, based on equity holdings, calculated based on arithmetic method
$\bar{g}$	=	The group where group g belongs in the prior grouping hierarchy level
$R_\text{Ø}^P$	=	The portfolio's total equity return, calculated based on equity holdings
$R_\text{Ø}^B$	=	The benchmark's total equity return, calculated based on equity holdings

The arithmetic method refers to simple subtractions and additions. For example, simple subtractions are used when comparing returns of the portfolio with the benchmark, as shown in the second term of formula [5]. Furthermore, active return in formula [7] is the simple addition of the total effects at various decision levels. These characteristics distinguish the arithmetic method from its geometric counterpart. The arithmetic method also serves as the foundation for the geometric method presented in the next section.

In the component calculation in equation [5], there are some terms that are similar to the basic BHB and BF models and many that are not. The BHB model is at the portfolio level, while formula [5], as its name indicates, is at the components level. In other words, formula [5] calculates how Asia and Europe, as components, each contribute toward the total regional weighting effect. Thus, it is more appropriate to compare it with the BF model.

Top-Down Approach for Single Period (Continued)

To fully understand the component calculation, let us first focus on the first multiplicative term of equation [5]. The BF model is founded on the concept of the weighting effect being the difference between the portfolio and benchmark weightings, and the first term of the component formula is essentially that difference. The dissimilarity between the BF model and the component formula stems from the latter being modified for a hierarchical decision-making structure. For example, an investment manager may first decide on regional weighting, followed by sector weighting and market-cap weighting, before security selection. The analysis is hierarchical because the weighting at each decision level is anchored upon the weighting of the prior decision. For example, let the portfolio's weighting in Asia be 60% and the benchmark's weighting in the same region be 30%, representing a double weighting. Further assume that there are two sectors in Asia, and the benchmark has half the weighting in each sector. Thus, each sector has a 15% benchmark weighting. As the portfolio has 60% in Asia, if it were to mimic the benchmark and place half its weighting in each of the two sectors, each sector would have a 30% portfolio weighting and look overweighted even though the portfolio mimics the benchmark's allocation. Therefore, one must not compare the portfolio weighting of the Asia region's cyclical sector directly with the weighting of the same sector in the benchmark. The fair comparison is to create an anchoring system like formula [5] where the benchmark's weighting in the Asia region's cyclical sector is scaled to the proportion between the portfolio's weighting in Asia and that of the benchmark. In this example, the benchmark's weighting in the Asian cyclical sector must be multiplied by 2 before it can be compared with the portfolio's weighting in the same sector because 2 is the result of 0.6 divided by 0.3.

The symbol $\overline{g}$ is the group that group g belongs to in the prior decision level of the hierarchy. Following the analogy of a family tree, $\overline{g}$ represents the parent of g. For example, the $\overline{g}$ term for the Asian cyclical sector represents the Asia region, as the Asian cyclical sector is part of the Asia region, and region is the decision level prior to sector. For simplicity, let us call it the "parent group" to group g. When $\overline{g} = \text{Ø}$, when the parent group is the total level, $w_{\text{Ø}}^{\mathrm{P}} = w_{\text{Ø}}^{\mathrm{B}} = 1$.

Top-Down Approach for Single Period (Continued)

Shifting focus to the second term of equation [5], this term is similar to the BF model. In order to adopt a hierarchical structure, the second term is transformed into the return differential between the group in question and its parent group. Thus, this term intuitively illustrates that a group is good if it outperforms the combined performance of all siblings, and vice versa. For example, if Europe's cyclical sector has a benchmark return of 8.40% while Europe has a benchmark return of 3.53%, the differential is 4.87%, a positive number demonstrating that this region's cyclical sector has outperformed other sectors in the region.

Formula [5] illustrates the same intuitive concepts in the BF article. It is good to overweight a group that has outperformed and underweight a group that has underperformed. It is bad to overweight a group that has underperformed and underweight a group that has outperformed.

Formula [6] shows that the effect of a parent group is the sum of the components of all of its children if the children are components of this decision. For example, when analyzing the sector-weighting effect, the sectors are components of the decision, so Asia's sector-weighting effect is the sum of the sector-weighting components of Asian cyclical and Asian defensive. The effect of a grandparent group is the sum of the effects of all of its children if the children's descendants are components of this decision. For example, when analyzing the selection effect, the securities are components of this decision. Thus, the Asian cyclical sector's selection effect is the sum of the selection effects of Asian cyclical large cap and Asian cyclical small cap, and these two are in turn sums of the selection components of the underlying constituent stocks.

The formulas for components and effects are universal to all grouping levels. When $n < M$, the result of the formula is referred to as a weighting effect. When $n = M$, the result of the formula is referred to as a selection effect. For example, $EA_{(1,2),4}$ is the selection (fourth decision) effect of the first region's second sector.

Active return in formula [7] is the value-add of equity securities, and it is the difference between the return of the equity portion of the portfolio and that of the equity portion of the benchmark. Expressed in attribution terms, the active return is the simple addition of the total effects at all decision levels. In other words, it is the sum of the total effects of all four decisions made in the portfolio: regional weighting, sector weighting, market-cap weighting, and security selection. This active return represents the value-add of the equity portion of the portfolio, and it is calculated based on equity holdings as of the beginning of the analysis period. Refer to the Appendix for the value-add of the total portfolio and residuals that account for the difference between actual and calculated returns.

Geometric Method

[8] $$R^{HL} = \begin{cases} R_{Ø}^{\mathrm{B}} & \text{if } L = 0 \\ \mathrm{EA}_{Ø,\mathrm{L}} + R^{H(L-1)} & \text{if } L > 0 \end{cases}$$

[9] $$CG_g = \frac{CA_g}{1 + R^{H|\bar{g}|}}$$

[10] $$EG_{g,n} = \begin{cases} \sum_{h \in \Omega_g} CG_h & \text{if } n = |g| + 1 \\ \sum_{h \in \Omega_g} EG_{h,n} & \text{if } n > |g| + 1 \end{cases}$$

[11] $$AG_{Ø} = \frac{1 + R_{Ø}^{\mathrm{P}}}{1 + R_{Ø}^{\mathrm{B}}} - 1 = \prod_{n=1}^{M} (1 + EG_{Ø,n}) - 1$$

Where:

R^{HL}	=	Return of the hybrid portfolio at level L
CG_g	=	Component attributable to group g, calculated based on geometric method
$EG_{g,n}$	=	Effect attributable to group g at decision level n, based on geometric method
$AG_{Ø}$	=	The portfolio's active return, based on equity holdings, calculated based on geometric method

Top-Down Approach for Single Period (Continued)

Equation [5] in the previous section presents the hierarchical anchoring system used in the component formula. The denominator of formula [9] in this section demonstrates another method of hierarchical anchoring, and it is facilitated by the use of the "hybrid" portfolio defined in formula [8]. The hybrid portfolio may look unfamiliar when presented in its concise presentation in formula [8], but it is based on the already familiar hierarchical anchoring system in equation [5]. Recall that in equation [5] the benchmark's weighting in the Asia region's cyclical sector is scaled to the proportion between the portfolio's weighting in Asia and that of the benchmark. The hybrid portfolio is similar to the benchmark portfolio in that the benchmark weighting in each sector is combined with the benchmark return in the sector, but the scaled benchmark weightings are used instead of the raw benchmark weightings. Mathematically the concise form in formula [8] yields the same result as combining the scaled benchmark weightings with benchmark returns. The concise form in formula [8] has the benefit of reusing numbers that are already calculated in the arithmetic method. Formula [8] shows that at the total level, no anchoring is required, and the hybrid portfolio is the same as the benchmark portfolio.

At levels other than the total level, the hybrid portfolio's return is the sum of the arithmetic total effect of this decision level and the hybrid return of the prior decision level. For example, the sector-level hybrid portfolio is the sum of the arithmetic total sector effect and the return of the regional hybrid portfolio.

The effect calculation in formula [10] needs no further explanation as it is similar to its arithmetic counterpart in formula [6]. The active return in formula [11] is also similar to its arithmetic counterpart in formula [7], but geometric operations are used instead of arithmetic operations. The active return is the geometric difference between the returns of the equity portion of the portfolio and the equity portion of the benchmark. Active return can also be computed by geometrically linking the total effects from all decision levels.

Top-Down Approach for Single Period (Continued)

Example

Following the example described earlier, below is a top-down investment process that consists of decision-making in the following order: regional weighting, sector weighting, market-cap weighting, and security selection.

Hierarchical Structure Illustration

	Region Wt	Sector Wt	Market Cap Wt	Sec Selection	Active Ret
Total	$EG_{Ø,1}$	$EG_{Ø,2}$	$EG_{Ø,3}$	$EG_{Ø,4}$	AG
Asia	$CG_{(1)}$	$EG_{(1),2}$	$EG_{(1),3}$	$EG_{(1),4}$	
Cyclical		$CG_{(1,1)}$	$EG_{(1,1),3}$	$EG_{(1,1),4}$	
Large Cap			$CG_{(1,1,1)}$	$EG_{(1,1,1),4}$	
Small Cap			$CG_{(1,1,2)}$	$EG_{(1,1,2),4}$	
Defensive		$CG_{(1,2)}$	$EG_{(1,2),3}$	$EG_{(1,2),4}$	
Large Cap			$CG_{(1,2,1)}$	$EG_{(1,2,1),4}$	
Small Cap			$CG_{(1,2,2)}$	$EG_{(1,2,2),4}$	
Europe	$CG_{(2)}$	$EG_{(2),2}$	$EG_{(2),3}$	$EG_{(2),4}$	
Cyclical		$CG_{(2,1)}$	$EG_{(2,1),3}$	$EG_{(2,1),4}$	
Large Cap			$CG_{(2,1,1)}$	$EG_{(2,1,1),4}$	
Small Cap			$CG_{(2,1,2)}$	$EG_{(2,1,2),4}$	
Defensive		$CG_{(2,2)}$	$EG_{(2,2),3}$	$EG_{(2,2),4}$	
Large Cap			$CG_{(2,2,1)}$	$EG_{(2,2,1),4}$	
Small Cap			$CG_{(2,2,2)}$	$EG_{(2,2,2),4}$	

Top-Down Approach for Single Period (Continued)

Attribution

	Weightings		Returns		Attribution				
	Portfolio	Bench	Portfolio	Bench	Region	Sector	Mkt Cap	Selection	Active Return
Total	1.00	1.00	0.0695	0.0529	0.0029	-0.0166	-0.0273	0.0588	0.0157
Asia	0.53	0.45	0.1304	0.0744	0.0016	0.0062	-0.0417	0.0658	
Cyclical	0.25	0.30	0.1400	0.0533		0.0021	-0.0167	0.0386	
Large Cap	0.15	0.05	0.1000	-0.0800			-0.0139	0.0267	
Small Cap	0.10	0.25	0.2000	0.0800			-0.0028	0.0119	
Defensive	0.28	0.15	0.1218	0.1167		0.0041	-0.0250	0.0272	
Large Cap	0.05	0.10	0.0700	0.1800			-0.0083	-0.0054	
Small Cap	0.23	0.05	0.1330	-0.0100			-0.0167	0.0326	
Europe	0.47	0.55	0.0009	0.0353	0.0013	-0.0228	0.0144	-0.0070	
Cyclical	0.12	0.35	0.0700	0.0840		-0.0083	-0.0029	0.0014	
Large Cap	0.00	0.10	0.1476	0.1476			-0.0021	0.0000	
Small Cap	0.12	0.25	0.0700	0.0586			-0.0008	0.0014	
Defensive	0.35	0.20	-0.0229	-0.0500		-0.0145	0.0173	-0.0084	
Large Cap	0.18	0.00	0.0500	0.0500			0.0173	0.0000	
Small Cap	0.17	0.20	-0.1000	-0.0500			0.0000	-0.0084	

First Decision: Regional Weighting

$$R^{H,0} = R^{\mathrm{B}}_{\text{Ø}} = (w^B_{(1)} \bullet R^B_{(1)} + w^B_{(2)} \bullet R^B_{(2)}) / w^{\mathrm{B}}_{\text{Ø}}$$
$$= (0.45 \bullet 0.0744 + 0.55 \bullet 0.0353) / 1.00 = 0.0529$$
$$CG_{(1)} = (w^P_{(1)} - w^P_{\text{Ø}} / w^B_{\text{Ø}} \bullet w^B_{(1)}) \bullet (R^B_{(1)} - R^B_{\text{Ø}}) / (1 + R^{H,0})$$
$$= (0.53 - 100/100 \bullet 0.45) \bullet (0.0744 - 0.0529) / (1 + 0.0529) = 0.0016$$
$$CG_{(2)} = (w^P_{(2)} - w^P_{\text{Ø}} / w^B_{\text{Ø}} \bullet w^B_{(2)}) \bullet (R^B_{(2)} - R^B_{\text{Ø}}) / (1 + R^{H,0})$$
$$= (0.47 - 100/100 \bullet 0.55) \bullet (0.0353 - 0.0529) / (1 + 0.0529) = 0.0013$$

Total regional weighting effect: $EG_{\text{Ø},1} = CG_{(1)} + CG_{(2)} = 0.0016 + 0.0013 = 0.0029$

Second Decision: Sector Weighting

$$R^{H,1} = EA_{Ø,1} + R^{H,0} = EG_{Ø,1} \bullet (1 + R^{H,0}) + R^{H,0}$$
$$= 0.0029 \bullet (1 + 0.0529) + 0.0529 = 0.0560$$
$$CG_{(1,1)} = (w^P_{(1,1)} - w^P_{(1)} / w^B_{(1)} \bullet w^B_{(1,1)}) \bullet (R^B_{(1,1)} - R^B_{(1)}) / (1 + R^{H,1})$$
$$= (0.25 - 0.53 / 0.45 \bullet 0.30) \bullet (0.0533 - 0.0744) / (1 + 0.0560) = 0.0021$$
$$CG_{(1,2)} = (w^P_{(1,2)} - w^P_{(1)} / w^B_{(1)} \bullet w^B_{(1,2)}) \bullet (R^B_{(1,2)} - R^B_{(1)}) / (1 + R^{H,1})$$
$$= (0.28 - 0.53 / 0.45 \bullet 0.15) \bullet (0.1167 - 0.0744) / (1 + 0.0560) = 0.0041$$
$$EG_{(1),2} = CG_{(1,1)} + CG_{(1,2)} = 0.0021 + 0.0041 = 0.0062$$
$$CG_{(2,1)} = (w^P_{(2,1)} - w^P_{(2)} / w^B_{(2)} \bullet w^B_{(2,1)}) \bullet (R^B_{(2,1)} - R^B_{(2)}) / (1 + R^{H,1})$$
$$= (0.12 - 0.47 / 0.55 \bullet 0.35) \bullet (0.0840 - 0.0353) / (1 + 0.0560) = -0.0083$$
$$CG_{(2,2)} = (w^P_{(2,2)} - w^P_{(2)} / w^B_{(2)} \bullet w^B_{(2,2)}) \bullet (R^B_{(2,2)} - R^B_{(2)}) / (1 + R^{H,1})$$
$$= (0.35 - 0.47 / 0.55 \bullet 0.20) \bullet (-0.0500 - 0.0353) / (1 + 0.0560) = -0.0145$$
$$EG_{(2),2} = CG_{(2,1)} + CG_{(2,2)} = (-0.0083) + (-0.0145) = -0.0228$$

Total sector weighting effect:

$$EG_{Ø,2} = EG_{(1),2} + EG_{(2),2} = 0.0062 + (-0.0228) = -0.0166$$

Third Decision: Market-Cap Weighting

$$R^{H,2} = EA_{Ø,2} + R^{H,1} = EG_{Ø,2} \bullet (1 + R^{H,1}) + R^{H,1}$$
$$= (-0.0166) \bullet (1 + 0.0560) + 0.0560 = 0.0385$$
$$CG_{(1,1,1)} = (w^P_{(1,1,1)} - w^P_{(1,1)} / w^B_{(1,1)} \bullet w^B_{(1,1,1)}) \bullet (R^B_{(1,1,1)} - R^B_{(1,1)}) / (1 + R^{H,2})$$
$$= (0.15 - 0.25 / 0.30 \bullet 0.05) \bullet (-0.0800 - 0.0533) / (1 + 0.0385) = -0.0139$$
$$CG_{(1,1,2)} = (w^P_{(1,1,2)} - w^P_{(1,1)} / w^B_{(1,1)} \bullet w^B_{(1,1,2)}) \bullet (R^B_{(1,1,2)} - R^B_{(1,1)}) / (1 + R^{H,2})$$
$$= (0.10 - 0.25 / 0.30 \bullet 0.25) \bullet (0.0800 - 0.0533) / (1 + 0.0385) = -0.0028$$
$$EG_{(1,1),3} = CG_{(1,1,1)} + CG_{(1,1,2)} = (-0.0139) + (-0.0028) = -0.0167$$
$$CG_{(1,2,1)} = (w^P_{(1,2,1)} - w^P_{(1,2)} / w^B_{(1,2)} \bullet w^B_{(1,2,1)}) \bullet (R^B_{(1,2,1)} - R^B_{(1,2)}) / (1 + R^{H,2})$$
$$= (0.05 - 0.28 / 0.15 \bullet 0.10) \bullet (0.1800 - 0.1167) / (1 + 0.0385) = -0.0083$$
$$CG_{(1,2,2)} = (w^P_{(1,2,2)} - w^P_{(1,2)} / w^B_{(1,2)} \bullet w^B_{(1,2,2)}) \bullet (R^B_{(1,2,2)} - R^B_{(1,2)}) / (1 + R^{H,2})$$
$$= (0.23 - 0.28 / 0.15 \bullet 0.05) \bullet (-0.0100 - 0.1167) / (1 + 0.0385) = -0.0167$$
$$EG_{(1,2),3} = CG_{(1,2,1)} + CG_{(1,2,2)} = (-0.0083) + (-0.0167) = -0.0250$$
$$EG_{(1),3} = EG_{(1,1),3} + EG_{(1,2),3} = (-0.0167) + (-0.0250) = -0.0417$$

Top-Down Approach for Single Period (Continued)

$$CG_{(2,1,1)} = (w^P_{(2,1,1)} - w^P_{(2,1)} / w^B_{(2,1)} \bullet w^B_{(2,1,1)}) \bullet (R^B_{(2,1,1)} - R^B_{(2,1)}) / (1 + R^{H,2})$$
$$= (0.00 - 0.12 / 0.35 \bullet 0.10) \bullet (0.1476 - 0.0840) / (1 + 0.0385) = -0.0021$$
$$CG_{(2,1,2)} = (w^P_{(2,1,2)} - w^P_{(2,1)} / w^B_{(2,1)} \bullet w^B_{(2,1,2)}) \bullet (R^B_{(2,1,2)} - R^B_{(2,1)}) / (1 + R^{H,2})$$
$$= (0.12 - 0.12 / 0.35 \bullet 0.25) \bullet (0.0586 - 0.0840) / (1 + 0.0385) = -0.0008$$
$$EG_{(2,1),3} = CG_{(2,1,1)} + CG_{(2,1,2)} = (-0.0021) + (-0.0008) = -0.0029$$
$$CG_{(2,2,1)} = (w^P_{(2,2,1)} - w^P_{(2,2)} / w^B_{(2,2)} \bullet w^B_{(2,2,1)}) \bullet (R^B_{(2,2,1)} - R^B_{(2,2)}) / (1 + R^{H,2})$$
$$= (0.18 - 0.35 / 0.20 \bullet 0.00) \bullet (0.0500 - (-0.0500)) / (1 + 0.0385) = 0.0173$$
$$CG_{(2,2,2)} = (w^P_{(2,2,2)} - w^P_{(2,2)} / w^B_{(2,2)} \bullet w^B_{(2,2,2)}) \bullet (R^B_{(2,2,2)} - R^B_{(2,2)}) / (1 + R^{H,2})$$
$$= (0.17 - 0.35 / 0.20 \bullet 0.20) \bullet (-0.0500 - (-0.0500)) / (1 + 0.0385) = 0$$
$$EG_{(2,2),3} = CG_{(2,2,1)} + CG_{(2,2,2)} = 0.0173 + 0 = 0.0173$$
$$EG_{(2),3} = EG_{(2,1),3} + EG_{(2,2),3} = (-0.0029) + 0.0173 = 0.0144$$

Total market-cap-weighting effect:

$$EG_{Ø,3} = EG_{(1),3} + EG_{(2),3} = (-0.0417) + 0.0144 = -0.0273$$

Fourth Decision: Security Selection

$$R^{H,3} = EA_{Ø,3} + R^{H,2} = EG_{Ø,3} \bullet (1 + R^{H,2}) + R^{H,2}$$
$$= (-0.0273) \bullet (1 + 0.0385) + 0.0385 = 0.0102$$

Instead of showing every stock in the portfolio and benchmark, let us show just one example and assume $w^P_{(1,1,1,1)} = 0.15$, $w^B_{(1,1,1,1)} = 0$, and $R^P_{(1,1,1,1)} = R^B_{(1,1,1,1)} = 0.1000$.

$$CG_{(1,1,1,1)} = (w^P_{(1,1,1,1)} - w^P_{(1,1,1)} / w^B_{(1,1,1)} \bullet w^B_{(1,1,1,1)}) \bullet (R^B_{(1,1,1,1)} - R^B_{(1,1,1)}) / (1 + R^{H,3})$$
$$= (0.15 - 0.15 / 0.05 \bullet 0) \bullet (0.1000 - (-0.0800) / (1 + 0.0102) = 0.0267$$

Further, assume that the total security-selection effect: $EG_{Ø,4} = 0.0588$.

Active Return

$$AG_{Ø} = (1 + EG_{Ø,1}) \bullet (1 + EG_{Ø,2}) \bullet (1 + EG_{Ø,3}) \bullet (1 + EG_{Ø,4}) - 1$$
$$= (1 + 0.0029) \bullet (1 - 0.0166) \bullet (1 - 0.0273) \bullet (1 + 0.0588) - 1 = 0.0157$$

Bottom-Up Approach for Single Period

Overview

As discussed in the Introduction, the bottom-up approach to portfolio attribution is most appropriately used when analyzing an investment manager with a bottom-up investment process that focuses on security selection. The weighting effect is secondary to the decision-making process. Unlike the top-down process, which may involve a series of weighting decisions, there is only one weighting effect in the bottom-up process.

This section addresses the bottom-up approach in a single-period attribution analysis. Multiperiod attribution is discussed in the last section of this document.

Arithmetic Method

[12] $$CA_g = (w_g^P \bullet \frac{w_{\bar{g}}^B}{w_{\bar{g}}^P} - w_g^B) \bullet (R_g^P - R_{\bar{g}}^B)$$

[13] $$EA_{g,n} = \begin{cases} \sum_{h \in \Omega_g} CA_h & \text{if } n = |\mathrm{g}| + 1 \\ \sum_{h \in \Omega_g} EA_{h,n} & \text{if } n > |\mathrm{g}| + 1 \end{cases}$$

[14] $$AA_{Ø} = R_{Ø}^{\mathrm{P}} - R_{Ø}^{\mathrm{B}} = EA_{Ø,1} + EA_{Ø,2}$$

Where:

CA_g	=	Component attributable to group g, calculated based on arithmetic method
$EA_{g,n}$	=	Effect attributable to group g at decision level n, based on arithmetic method
$AA_{Ø}$	=	The equity portfolio's active return, based on equity holdings, calculated based on arithmetic method
$\bar{g}$	=	The group where group g belongs in the prior grouping hierarchy level
n	=	Decision level, where $n = 1$ is the weighting decision and $n = 2$ is the security-selection decision
Ø	=	The total level, which is the total equity

These formulas are similar to their counterparts in the "Top-Down Approach" section of this document. The component formula in equation [12] demonstrates a hierarchical anchoring structure that is similar to that of its top-down counterpart in equation [5]. In the case of formula [12], it is the portfolio weighting of a group that is scaled to the proportion between the benchmark's weighting and the portfolio's weighting in the parent group. Once scaled, the portfolio weighting can be fairly compared with the benchmark weighting. In other words, when evaluating the stock-selection component of a particular stock, one should not compare the portfolio's weighting in the stock directly with the benchmark's weighting in the same stock. One must scale the portfolio's weighting in this stock by the proportion between the benchmark's weighting in the sector and the portfolio's weighting in the sector, assuming the investment manager groups stocks by sector.

To accompany this anchoring system, it is the portfolio's return in the security that is compared with the benchmark's return in the sector in the second term of formula [12]. Similarly, when evaluating a sector, it is the portfolio's return in the sector that is compared with the benchmark's total return, and this is consistent with incorporating the interaction term of the BHB model into the weighting effect in a bottom-up approach.

The effect formula in equation [13] is intentionally written to be the same as its top-down counterpart in equation [6], a concept that is already familiar. In order to achieve this, $n=1$ is set to denote the weighting decision and $n=2$ the security-selection decision, even though security selection is the primary decision. This order is more intuitive as it matches the grouping hierarchy structure where $|g|=1$ represents the sector and $|g|=2$ the security. Formula [13] shows that the effect of a parent group is the sum of the components of all of its children if the children are components of this decision. For example, when analyzing the selection effect, the securities are components of the decision, so the cyclical sector's selection effect is the sum of the selection components of all stocks in the sector. The effect of a grandparent group is the sum of the effects of all of its children. For example, the total equity portfolio's selection effect is the sum of the selection effects of cyclical and defensive sectors, and these two are in turn sums of selection components of the underlying constituent stocks.

Active return in formula [14] is the same as its top-down counterpart in equation [7], but only two decisions are involved: weighting and selection. Active return is the value-add of the portfolio above the benchmark, and it is the difference between the return of the equity portion of the portfolio and that of the equity portion of the benchmark. Expressed in attribution terms, active return is the simple addition of the total weighting effect and the total selection effect. This active return represents the value-add of the equity portion of the portfolio. Refer to the Appendix for the value-add of the total portfolio.

Geometric Method

[15] $$CG_g = \begin{cases} \dfrac{CA_g}{1 + EA_{Ø,2} + R_Ø^B} & \text{if } |g| = 1 \\ \dfrac{CA_g}{1 + R_Ø^B} & \text{if } |g| = 2 \end{cases}$$

[16] $$EG_{g,n} = \begin{cases} \sum_{h \in \Omega_g} CG_h & \text{if } n = |\mathrm{g}| + 1 \\ \sum_{h \in \Omega_g} EG_{h,n} & \text{if } n > |\mathrm{g}| + 1 \end{cases}$$

[17] $$AG_Ø = \frac{1 + R_Ø^{\mathrm{P}}}{1 + R_Ø^{\mathrm{B}}} - 1 = (1 + EG_{Ø,1}) \bullet (1 + EG_{Ø,2}) - 1$$

Where:

CG_g	=	Component attributable to group g, calculated based on geometric method
$EG_{g,n}$	=	Effect attributable to group g at decision level n, based on geometric method
$AG_Ø$	=	The equity portfolio's active return, based on equity holdings, calculated based on geometric method

Bottom-Up Approach for Single Period (continued)

The geometric component formula in equation [15] is similar to its top-down counterpart in formulas [8] and [9]. While the top-down approach allows multiple decisions and is better presented with two formulas, the bottom-up approach only requires one formula as there are only two decisions. Similar to equation [9], the component formula in equation [15] shows the use of the "hybrid" portfolio in the denominator to facilitate hierarchical anchoring. This anchoring system is similar to the one used in equation [12] for the arithmetic component calculation. Recall that in equation [12] the portfolio's weighting in a stock is scaled to the proportion between the benchmark's weighting in the sector and that of the portfolio. The hybrid portfolio is similar to the actual portfolio in that the portfolio's weighting in each stock is combined with the portfolio return in each stock, but the scaled portfolio weightings are used instead of the raw portfolio weightings. Mathematically the concise form in the denominator of equation [15] yields the same result as combining the scaled portfolio weightings with portfolio returns, and the concise form has the benefit of reusing numbers that are already calculated in the arithmetic method.

The effect formula in equation [16] is the same as its arithmetic counterpart in formula [13] and its top-down counterpart in formula [10]. Formula [16] shows that the effect of a parent group is the sum of the components of all of its children if the children are components of this decision. The effect of a grandparent group is the sum of the effects of all of its children if the children's descendants are components of this decision.

The active return calculation in formula [17] is essentially the same as its top-down counterpart in formula [11], demonstrating that active return is achieved by taking the geometric difference between the portfolio's equity return and the benchmark's equity return or by geometrically linking the total weighting and selection effects.

Three-Factor Approach for Single Period

Overview

The three-factor approach decomposes the active return into weighting effect, selection effect, and interaction. Unlike the top-down and bottom-up approaches presented in the previous sections, the three-factor model takes an agnostic view regarding the order of decision-making in the investment process. Thus, there is no distinction between primary and secondary effects. This approach to performance attribution is most appropriately used when one seeks purity in both weighting and selection effects and isolates the interaction between these two decisions into its own term. As stated in the Introduction, the interaction term does not represent an explicit decision of the investment manager, and Morningstar believes that it is a better practice to use the top-down and bottom-up approaches where the interaction term is embedded into the secondary effect of the investment process.

This section addresses the three-factor approach in a single-period attribution analysis. Multiperiod attribution is discussed in the last section of this document.

Three-Factor Approach for Single Period (Continued)

Arithmetic Method

[18] $$CA_g = \begin{cases} (w_g^P \bullet \dfrac{w_{\bar{g}}^B}{w_{\bar{g}}^P} - w_g^B) \bullet (R_g^B - R_{\bar{g}}^B) & \text{if } n = |\text{g}| \\ \sum\limits_{h \in \Omega_g} CA_h & \text{if } n > |\text{g}| \end{cases}$$

[19] $$EA_{Ø,n} = \sum_{h \in \Omega_Ø} CA_h$$

[20] $$IA_g = \begin{cases} (w_g^P - w_g^B) \bullet (R_g^P - R_g^B) & \text{if } |\text{g}| = 1 \\ \sum\limits_{h \in \Omega_\text{g}} IA_h & \text{if } |\text{g}| = 0 \end{cases}$$

[21] $$AA_Ø = R_Ø^\text{P} - R_Ø^\text{B} = EA_{Ø,1} + EA_{Ø,2} + IA_Ø$$

Where:

CA_g	=	Component attributable to group g , calculated based on arithmetic method
$EA_{Ø,n}$	=	Effect attributable to the total equity portfolio at decision level n , based on arithmetic method
IA_g	=	Interaction attributable to group g , based on arithmetic method
$AA_Ø$	=	The equity portfolio's active return, based on equity holdings, calculated based on arithmetic method
$\bar{g}$	=	The group where group g belongs in the prior grouping hierarchy level
n	=	Decision level, where $n=1$ is the weighting decision, and $n=2$ is the security-selection decision
Ø	=	The total level, which is the total equity

Three-Factor Approach for Single Period (Continued)

These formulas are similar to their counterparts in the "Bottom-Up Approach" section of this document. The component formula in equation [18] demonstrates a hierarchical anchoring structure similar to that of its bottom-up counterpart in equation [12]. In formula [18], the portfolio weighting of a group is scaled to the proportion between the benchmark's weighting and the portfolio's weighting in the parent group. Once scaled, the portfolio weighting can be compared fairly with the benchmark weighting. In other words, when evaluating the stock-selection component of a particular stock, one should not compare the portfolio's weighting in the stock directly with the benchmark's weighting in the same stock. One must scale the portfolio's weighting in this stock by the proportion between the benchmark's weighting in the sector and the portfolio's weighting in the sector, assuming the investment manager groups stocks by sector.

To accompany this anchoring system, it is the benchmark's return in the security that is compared with the benchmark's return in the sector in the second term of formula [18]. Similarly, when evaluating a sector, it is the benchmark's return in the sector that is compared with the benchmark's total return.

Formula [18] has a different form when $n > |\mathrm{g}|$. These symbols are intentionally written to be similar to their counterparts in the top-down and bottom-up approaches. In order to achieve this, $n = 1$ is set to denote the weighting decision and $n = 2$ the security-selection decision, even though there is not a distinction between primary and secondary effects in the three-factor approach. This order is more intuitive as it matches the grouping hierarchy structure where $|g| = 1$ represents the sector and $|g| = 2$ the security. However, there is a significant difference between the three-factor model and its top-down and bottom-up counterparts when it comes to the determination of an effect versus a component. In the top-down and bottom-up approaches, a term is an effect when $n > |\mathrm{g}|$. However, in the three-factor model only the total equity level is considered an effect. Thus, the second portion of formula [18] is for subtotals where $n > |\mathrm{g}|$. For example, the cyclical sector's selection component effect is the sum of selection components of all stocks in the sector.

Formula [19] shows that the effect of the total equity level is the sum of the components of all of its children. For example, the total equity portfolio's selection effect is the sum of the selection effects of the cyclical and defensive sectors, and these two are in turn sums of the selection components of the underlying constituent stocks.

Three-Factor Approach for Single Period (Continued)

The interaction term in formula [20] confirms that it is indeed the interaction between the weighting and selection decisions, as it is the cross-product of active weighting management and active selection decision. The formula is the same as the one in the BHB article, but it is deconstructed into two equations here for applications at the sector and total equity portfolio levels. At the sector level, the interaction term is the product between a particular sector's weighting relative to the benchmark and its relative return. A positive interaction term in a particular sector demonstrates that the investment manager is successful in overweighting the sector when active management added value, or underweighting a sector when active management subtracted value. In contrast, a negative interaction term in a particular sector implies that the investment manager has made an unsuccessful decision in overweighting the sector when active management subtracted value, or underweighting the sector when active management added value.

Note that when calculating the interaction term, it does not matter how the sector performs compared with the overall benchmark; what matters is whether the portfolio's return in the sector is better than that of the benchmark in the same sector. Thus, counterintuitively, it is possible for the interaction term to be positive even if the sector has poor weighting and selection attribution results. This happens when the portfolio is underweight in a sector where active management is poor but the sector still outperforms the overall benchmark. For example, let us assume that the portfolio returns 5% in the cyclical sector while the benchmark returns 8%, and the benchmark's overall return is 2%. In this case, if the portfolio is underweight in the cyclical sector, the cyclical sector's component of the weighting effect would be negative for having an underweighting in a sector that outperforms the benchmark (8% versus 2%). The sector's selection effect is negative as the portfolio underperforms the benchmark in the sector (5% versus 8%). However, the sector's interaction term is positive even though weighting and selection are both poor, as the portfolio is underweight in a sector where active management underperforms (5% versus 8%).

Three-Factor Approach for Single Period (Continued)

Active return in formula [21] is similar to its top-down and bottom-up counterparts in equations [7] and [14], but the interaction term must be included. Active return is the portfolio's value-add above the benchmark, and it is the difference between the return of the equity portion of the portfolio and that of the equity portion of the benchmark. Expressed in attribution terms, the active return is the simple addition of the total weighting effect, the total selection effect, and the interaction term. This active return represents the value-add of the equity portion of the portfolio. Refer to the Appendix for the value-add of the total portfolio.

Three-Factor Approach for Single Period (Continued)

Geometric Method

[22] $$CG_g = \frac{CA_g}{1 + R_{Ø}^{B}}$$

[23] $$EG_{Ø,n} = \sum_{h \in \Omega_{Ø}} CG_h$$

[24] $$IG_g = \begin{cases} \dfrac{IA_g}{IA_{Ø}} \bullet \left(\dfrac{1 + R_{Ø}^{\mathrm{P}}}{1 + R_{Ø}^{\mathrm{B}}} \bullet \dfrac{1}{(1 + EG_{Ø,1}) \bullet (1 + EG_{Ø,2})} - 1 \right) & \text{if } |\mathrm{g}| = 1 \\ \sum_{h \in \Omega_{\mathrm{g}}} IG_h & \text{if } |\mathrm{g}| = 0 \end{cases}$$

[25] $$AG_{Ø} = \frac{1 + R_{Ø}^{\mathrm{P}}}{1 + R_{Ø}^{\mathrm{B}}} - 1 = (1 + EG_{Ø,1}) \bullet (1 + EG_{Ø,2}) \bullet (1 + IG_{Ø}) - 1$$

Where:

CG_g	=	Component attributable to group g , calculated based on geometric method
$EG_{Ø,n}$	=	Effect attributable to the total equity portfolio at decision level n , based on geometric method
IG_g	=	Interaction attributable to group g , based on geometric method
$AG_{Ø}$	=	The equity portfolio's active return, based on equity holdings, calculated based on geometric method

The geometric component formula in equation [22] is a simplified version of its top-down and bottom-up counterparts in formulas [9] and [15]. Similar to equations [9] and [15], the component formula in equation [22] shows the use of the "hybrid" portfolio in the denominator to facilitate anchoring. However, as the three-factor model is agnostic on the order of decision-making, both weighting and selection are anchored on the total equity benchmark just as primary decisions are anchored in the top-down and bottom-up geometric calculations.

Three-Factor Approach for Single Period (Continued)

The effect formula in equation [23] is the same as its arithmetic counterpart in formula [19]. Formula [23] shows that the effect of the total equity is the sum of the components of all of its children. As stated in the "Multiple Period Analysis" section, only effects can be geometrically compounded over time. Therefore, when running a three-factor geometric model in a multiperiod setting, only the attribution results at the total equity level will be presented, and no details will be provided at levels below the total equity such as sector or security level.

The interaction term in formula [24] is not immediately intuitive. There is not an intuitive explanation for the anchoring process used in transforming the arithmetic interaction term to its geometric format. Therefore, the anchor is obtained through backward engineering, knowing that the excess return is the result of geometrically linking the geometric weighting effect, the geometric selection effect, and the geometric interaction term. Thus, one can infer the geometric interaction term and solve for the multiplier needed in converting the arithmetic interaction term into its geometric format.

The active return calculation in formula [25] is essentially the same as its top-down and bottom-up counterparts in formulas [11] and [17], but the equation has been expanded to accommodate the interaction term. The formula demonstrates that active return is achieved by taking the geometric difference between the portfolio's equity return and the benchmark's equity return, or by geometrically linking the total weighting effect, the total selection effect, and the interaction term.

Multiple-Period Analysis

Overview

The previous sections of this document demonstrate how effects and components are calculated for each single holding period. A holding period is determined by portfolio holdings update, benchmark holdings update, and month-end. For example, if portfolio holdings are available on Jan. 31, 2008, March 15, 2008, and April 30, 2008, and assuming that benchmark holdings are available at quarter-ends, the period between Feb. 1, 2008, and Feb. 28, 2008, represents the first holding period, the period between March 1, 2008, and March 15, 2008, is the second holding period, the period between March 16, 2008, and March 31, 2008, is the third period, and the period between April 1, 2008, and April 30, 2008, is the fourth period. When applying the formulas in the previous sections, weightings are taken from the beginning of the period, and returns are based on the entire holding period. For example, when analyzing the first holding period, weightings are based on Jan. 31, 2008, and returns are from Feb. 1, 2008, to March 31, 2008.

It is often desirable to perform an analysis that spans over several portfolio holding dates, for example, from Feb. 1, 2008, to June 30, 2008. Although one might think of treating this as a single period, that is, taking the weightings as of Jan. 31, 2008, and applying them to returns from Feb. 1, 2008, to June 30, 2008, valuable information could be lost. Portfolio constituents and their weightings might have changed between Jan. 31, 2008, and March 31, 2008, due to buys, sells, adds, trims, corporate actions, and so on. For the most meaningful analysis, portfolio holdings should be updated frequently, especially for higher-turnover portfolios. Frequent portfolio holding updates create multiple single periods, and the following sections demonstrate how these single-period attribution results can be accumulated into an overall multiperiod outcome.

Multiperiod attribution effects consist of accumulating single-period results. Similar to a single-period analysis, results can be calculated using the arithmetic method or the geometric method. Use the multiperiod arithmetic method to accumulate single-period arithmetic attribution results, and use the multiperiod geometric method to link single-period geometric results. These multiperiod methods apply to attribution results from both the top-down and the bottom-up approaches.

Multiperiod Geometric Method

The geometric method is the method recommended by Morningstar. The geometric method has the merit of being theoretically and mathematically sound. As stated in the Introduction, it is important to distinguish between effects and components when performing an attribution analysis. An effect measures the impact of a particular investment decision. An effect can be broken down into several components (for example, individual sectors such as cyclical) that provide insight on each piece of an overall decision, but each piece in isolation cannot represent the impact of decision-making. Therefore, theoretically, multiperiod linking is only applicable to an effect and not a component. From a mathematical viewpoint, accumulating components over time either by adding or compounding, and adding them back together either by simple summation or geometric linking, does not equal the active return.

Use the following formulas to link single-period geometric attribution effects into multiperiod results:

[26] $$EG_{g,n,T,Cum} = \prod_{t=1}^{T}(1 + EG_{g,n,t}) - 1$$

[27] $$AG_{Ø,T,Cum} = \frac{1 + R^{P}_{Ø,T,Cum}}{1 + R^{B}_{Ø,T,Cum}} - 1 = \prod_{t=1}^{T}(1 + AG_{Ø,t}) - 1 = \prod_{n=1}^{M} EG_{Ø,n,T,Cum}$$

Where:

$EG_{g,n,T,Cum}$	=	Cumulative effect for group g decision level n, calculated based on geometric method, cumulative from single holding periods 1 to T
$AG_{Ø,T,Cum}$	=	Cumulative active return of the portfolio, calculated based on geometric method, cumulative from single holding periods 1 to T
$EG_{g,n,t}$	=	Effect attributable to group g at decision level n, calculated based on geometric method, for single period t
$R^{P}_{Ø,T,Cum}$	=	The portfolio's return for the total level (total equity portfolio), cumulative from single holding period from 1 to T
$R^{B}_{Ø,T,Cum}$	=	The benchmark's return for the total level (total equity portfolio), cumulative from single holding period from 1 to T
$AG_{Ø,t}$	=	Active return of the portfolio, calculated based on geometric method, for single period t
M	=	The level that represents the security level, that is, the last grouping hierarchy

Multiple-Period Analysis (Continued)

Use the following formulas to annualize multiperiod results:

[28] $$EG_{g,n,T,Ann} = (1 + EG_{g,n,T,Cum})^{\frac{y}{m}} - 1,$$

[29] $$AG_{Ø,T,Ann} = (1 + AG_{Ø,T,Cum})^{\frac{y}{m}} - 1,$$

where

$EG_{g,n,T,Ann}$	=	Annualized effect for group g decision level n , calculated based on geometric method, over the time period from 1 to T
$AG_{Ø,T,Ann}$	=	Annualized active return of the portfolio, calculated based on geometric method, over the time period from 1 to T
$EG_{g,n,T,Cum}$	=	Cumulative effect for group g decision level n , calculated based on geometric method, cumulative from single holding periods 1 to T
$AG_{Ø,T,Cum}$	=	Cumulative active return of the portfolio, calculated based on geometric method, cumulative from single holding periods 1 to T
y	=	The number of periods in a year; for example, it is 12 when data are in monthly frequency
m	=	The total number of periods; for example, it is 40 when the entire time period spans over 40 months

Multiperiod Arithmetic Method

As stated in the previous section, the geometric method is the one recommended by Morningstar for its theoretical and mathematical soundness. Multiperiod linking is only applicable to an effect and not a component. As components provide additional insight, several methodologies have emerged to accumulate components over multiple time periods. The word "accumulate" is a more appropriate term than the word "link" as components and effects are added over time rather than geometrically compounded. These alternative methodologies are commonly called triple-sum, as the cumulative active return (excess return over benchmark) over multiple periods is the sum of components in all groups (for example, sectors), decisions (weighting versus selection), and time periods. As adding components over time does not equal the cumulative active return, additional mathematical "smoothing" is applied to make them match. Mathematical smoothing is where formulas and philosophies differ among various alternative methodologies. It is important to make sure the choice of method does not significantly distort the reality--for example, altering the relative results of components and effects or causing a detractor to appear as a contributor or vice versa.

This document presents one of several arithmetic methodologies, the Modified Frongello methodology.[3] The use of the arithmetic method in the Modified Frongello methodology should not be interpreted as an endorsement from Morningstar.

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[3] Frongello, Andrew Scott Bay, "Readers' Reflections," *Journal of Performance Measurement*, Winter 2002/2003, pp. 7-11. This methodology is a modified version of the authors' original work in "Linking Single Period Attribution Results," *Journal of Performance Measurement*, Spring 2002, pp.10-22.

Multiple-Period Analysis (Continued)

Use the following formulas to accumulate single-period arithmetic attribution components and effects into multiperiod results:

[30]

$$CA_{g,T,Cum} = \frac{(2 + R^B_{Ø,T} + R^P_{Ø,T})}{2} \bullet CA_{g,T-1,Cum} + \frac{(2 + R^B_{Ø,T\text{-}1,Cum} + R^P_{Ø,T\text{-}1,Cum})}{2} \bullet CA_{g,T}$$

[31]

$$EA_{g,n,T,Cum} = \frac{(2 + R^B_{Ø,T} + R^P_{Ø,T})}{2} \bullet EA_{g,n,T-1,Cum} + \frac{(2 + R^B_{Ø,T\text{-}1,Cum} + R^P_{Ø,T\text{-}1,Cum})}{2} \bullet EA_{g,n,T}$$

[32] $$AA_{Ø,T,Cum} = R^P_{Ø,T,Cum} - R^B_{Ø,T,Cum} = \sum_{n=1}^{M} EA_{Ø,n,T,Cum}$$

Where:

$CA_{g,T,Cum}$	=	Cumulative component attributable to group g, calculated based on arithmetic method, cumulative from single holding periods 1 to T
$EA_{g,n,T,Cum}$	=	Cumulative effect attributable to group g at decision level *n*, calculated based on arithmetic method, cumulative from single holding periods 1 to T
$AA_{Ø,T,Cum}$	=	The portfolio's cumulative active return, based on arithmetic method, cumulative from periods 1 to T
$R^B_{Ø,T}$	=	The benchmark's return for the total level (total equity portfolio), at single holding period T
$R^P_{Ø,T}$	=	The portfolio's return for the total level (total equity portfolio), at single holding period T
$R^B_{Ø,T,Cum}$	=	The benchmark's return for the total level (total equity portfolio), cumulative from periods 1 to T
$R^P_{Ø,T,Cum}$	=	The portfolio's return for the total level (total equity portfolio), cumulative from periods 1 to T
$CA_{g,T}$	=	Component at single holding period *T* for group g, based on arithmetic method
$EA_{g,n,T}$	=	Effect at single holding period *T* for group g decision level n, based on arithmetic method
M	=	The level that represents the security level, that is, the last grouping hierarchy

Multiple-Period Analysis (Continued)

Note:

- At period T=1, $CA_{g,T,Cum} = CA_g$ and $EA_{g,n,T,Cum} = EA_{g,n}$, and these terms are defined in the "Single-Period" sections.

Use the following formulas to annualize multiperiod results:

[33] $$CA_{g,T,Ann} = CA_{g,T,Cum} \bullet \frac{y}{m}$$

[34] $$EA_{g,n,T,Ann} = EA_{g,n,T,Cum} \bullet \frac{y}{m}$$

[35] $$AA_{Ø,T,Ann} = AA_{Ø,T,Cum} \bullet \frac{y}{m}$$

Where:

$CA_{g,T,Ann}$	=	Annualized component attributable to group g, calculated based on arithmetic method, over the time period from 1 to T
$EA_{g,n,T,Ann}$	=	Annualized effect attributable to group g at decision level n, calculated based on arithmetic method, over the time period from 1 to T
$AA_{Ø,T,Ann}$	=	The portfolio's annualized active return, calculated based on arithmetic method, over the time period from 1 to T
$CA_{g,T,Cum}$	=	Cumulative component attributable to group g, calculated based on arithmetic method, cumulative from single holding periods 1 to T
$EA_{g,n,T,Cum}$	=	Cumulative effect attributable to group g at decision level n, calculated based on arithmetic method, cumulative from single holding periods 1 to T
$AA_{Ø,T,Cum}$	=	The portfolio's cumulative active return, based on arithmetic method, cumulative from periods 1 to T
y	=	The number of periods in a year; for example, it is 12 when data are in monthly frequency
m	=	The total number of periods; for example, it is 40 when the entire time period spans over 40 months

Multiple-Period Analysis (Continued)

- Unlike the geometric method, in which frequency conversion such as annualizing is clearly defined, there is not a defined formula for annualizing an arithmetic method; therefore, (y/m) is adopted with the end goal of preserving the arithmetic method's additive property across groups and types of attribution effects.

Appendix A: Contribution

Overview
Contribution is a topic that is often associated with attribution. Contribution is an absolute analysis that multiplies the absolute weighting in a sector or security by the absolute return, so an investment with mediocre performance may have a large contribution simply because a large amount of money is invested in it. Attribution, on the other hand, shows relative weighting and relative returns. A good result can come only from overweighting an outperforming investment or underweighting an underperforming one, a better measure of an investment manager's skill. As contribution is an absolute analysis and attribution a relative analysis, they complement each other.

Single Period
In a single-period analysis, the contribution is defined as

[36] $$C_g^P = \begin{cases} w_g^P \bullet R_g^P & \text{if } |\mathrm{g}| = M \\ \sum_{h \in \Omega_g} C_h^P & \text{if } |\mathrm{g}| < M \end{cases}$$

[37] $$C_g^B = \begin{cases} w_g^B \bullet R_g^B & \text{if } |\mathrm{g}| = M \\ \sum_{h \in \Omega_g} C_h^B & \text{if } |\mathrm{g}| < M \end{cases}$$

where

C_g^P	=	Contribution toward the portfolio associated with group g
C_g^B	=	Contribution toward the benchmark is associated with group g
M	=	The level that represents the security level; that is, the last grouping hierarchy

Appendix A: Contribution (Continued)

Multiple Period

Use the following formulas to link single-period contribution into multiperiod results:

[38] $$C^P_{g,T,Cum} = \sum_{t=1}^{T}(1+R^P_{\varnothing,t-1,Cum}) \bullet w^P_{g,t} \bullet R^P_{g,t} = \sum_{t=1}^{T}(1+R^P_{\varnothing,t-1,Cum}) \bullet C^P_{g,t}$$

[39] $$C^B_{g,T,Cum} = \sum_{t=1}^{T}(1+R^B_{\varnothing,t-1,Cum}) \bullet w^B_{g,t} \bullet R^B_{g,t} = \sum_{t=1}^{T}(1+R^B_{\varnothing,t-1,Cum}) \bullet C^B_{g,t}$$

[40] $$C^P_{g,T,Ann} = (1+C^P_{g,T,Cum})^{\frac{y}{m}} - 1$$

[41] $$C^B_{g,T,Ann} = (1+C^B_{g,T,Cum})^{\frac{y}{m}} - 1$$

Where:

$C^P_{g,T,Cum}$	=	Cumulative contribution toward the portfolio associated with group g, cumulative from single holding periods 1 to T
$C^B_{g,T,Cum}$	=	Cumulative contribution toward the benchmark associated with group g, cumulative from single holding periods 1 to T
$C^P_{g,T,Ann}$	=	Annualized contribution toward the portfolio associated with group g, cumulative from single holding periods 1 to T
$C^B_{g,T,Ann}$	=	Annualized contribution toward the portfolio associated with group g, cumulative from single holding periods 1 to T
$R^P_{\varnothing,t-1,Cum}$	=	The portfolio's return for the total level, cumulative from periods 1 to $t-1$
$R^B_{\varnothing,t-1,Cum}$	=	The benchmark's return for the total level, cumulative from periods 1 to $t-1$

Appendix A: Contribution (Continued)

At segment levels--for example, a particular sector or stock--the multiperiod contribution is not the geometric compounding of single-period contribution figures. Weighting changes may be due to transactions that cause the capital base for the segment to change. Therefore, at the beginning of each period it is necessary to compute the segment's capital base by multiplying the total portfolio's wealth by the segment's weighting, before applying the base to the period's contribution. The total portfolio's wealth at the beginning of the period is represented by 1 plus the cumulative portfolio return up to that time; in other words, it is the growth of $1. Expressed another way, a segment's multiperiod contribution is the sum of this segment's dollar contributions from every period, assuming a $1 initial investment in the total portfolio. When contributions are expressed in cumulative terms, segment contributions sum to that of the total portfolio. However, once annualized, these numbers no longer add up.

Appendix B: Transaction-Based Attribution

Overview

A conventional performance attribution analysis is holding-based. This is based on the assumption that portfolio holdings at the beginning of a single period are held until the end of the period with no transactions in between. In reality, transactions occur, and the closest a holding-based attribution analysis can reflect reality is by updating portfolio holdings at a daily frequency. However, a daily holding-based attribution analysis ignores intraday trades and may lead to an unexplained residual in the analysis--in other words, a gap between the actual return of the portfolio and the portfolio return computed by the weighted average of individual security returns. A transaction-based attribution analysis has the advantage of capturing the impact of these intraday trades and minimizing the unexplained residual. Although the impact of intraday trading can be implied by the residual in a daily holdings-based analysis, a transaction-based analysis has the added benefit of itemizing every trade's value-add.

In a transaction-based attribution analysis, the transaction effect captures the portion of performance attributable to trade execution and can only be calculated if transaction information is provided. Although traders may use other measures to evaluate their execution, in an attribution analysis the value-add of trade execution is defined as the difference between the transaction price and the end-of-day price. This is because the latter is the pricing used in performance measurement. Securities may be deposited or withdrawn in kind from a portfolio, and in such cases the transaction effect is calculated based on the deposit or withdrawal price instead of the trade price.

Single Period
One more rule is required in defining a single period for transaction-based attribution, in addition to the single-period rules applicable to holding-based attribution stated in the "Multiple-Period Analysis" section of this document. The day of the transaction must be its own single period. For example, if there is a transaction on Jan. 5, the previous single period ends on Jan. 4, Jan. 5 is its own single period, and the next single period starts on Jan. 6.

The formulas for transaction-based weighting, return, and contribution are

[42]
$$\tilde{w}_g^P = \begin{cases} \dfrac{BMV_g^P + TC_g^P}{BMV_{\text{Ø}}^P} & \text{if } |g| = M \\ \sum\limits_{h \in \Omega_g} \tilde{w}_h^P & \text{if } 0 < |g| < M \end{cases}$$

[43]
$$\tilde{R}_g^P = \begin{cases} R_g^P & \text{if } |g| = M \text{ and } TC_g = TW_g = 0 \\ \dfrac{EMV_g^P - TC_g^P + TW_g^P - BMV_g^P}{BMV_g^P + TC_g^P} & \text{if } |g| = M \text{ and } (TC_g \neq 0 \text{ or } TW_g \neq 0) \\ \dfrac{\sum\limits_{h \in \Omega_g} \tilde{w}_h^P \bullet \tilde{R}_h^P}{\tilde{w}_g^P} & \text{if } 0 < |g| < M \\ \sum\limits_{h \in \Omega_g} \tilde{w}_h^P \bullet \tilde{R}_h^P & \text{if } |g| = 0 \end{cases}$$

[44]
$$\tilde{C}_g^P = \begin{cases} \tilde{w}_g^P \bullet \tilde{R}_g^P & \text{if } |g| = M \\ \sum\limits_{h \in \Omega_g} \tilde{C}_h^P & \text{if } |g| < M \end{cases}$$

Appendix B: Transaction-Based Attribution (Continued)

where

$\tilde{w}_g^P$	=	Portfolio's effective weighting for the purpose of transaction-based attribution calculation, for group g
$\tilde{R}_g^P$	=	Portfolio's transaction-based return for group g
$\tilde{C}_g^P$	=	Portfolio's transaction-based contribution to return for group g
BMV_g^P		Portfolio's beginning market value for group g, where $g = \varnothing$ represents the total portfolio
TC_g^P		Portfolio's total deposit/contribution (sum of all purchases and transfers in) for group g
EMV_g^P		Portfolio's ending market value for group g
TW_g^P		Portfolio's total withdrawal (sum of all sells and transfers out) for group g

Note:

- The portfolio's effective weighting, $\tilde{w}_g^P$, is for calculation purposes only and is not displayed.

The formulas for transaction-based attribution measures are

[45] $$TA_g = \begin{cases} \tilde{w}_g^P \bullet \tilde{R}_g^P - w_g^P \bullet R_g^P & \text{if } |g| = M \\ \sum_{h \in \Omega_g} TA_h & \text{if } |g| < M \end{cases}$$

[46] $$AA_{Ø} = \tilde{R}_{Ø}^P - R_{Ø}^B = \sum_{n=1}^{M} EA_{Ø,\mathrm{n}} + IA_{Ø} + TA_{Ø}$$

[47] $$TG_g = \begin{cases} \dfrac{TA_g}{TA_{Ø}} \bullet \left(\dfrac{1 + \tilde{R}_{Ø}^P}{1 + R_{Ø}^P} - 1 \right) & \text{if } |g| = M \\ \sum_{h \in \Omega_g} TG_h & \text{if } |g| < M \end{cases}$$

[48] $$AG_{Ø} = \frac{1 + \tilde{R}_{Ø}^P}{1 + R_{Ø}^B} - 1 = \prod_{n=1}^{M} \left(1 + EG_{Ø,\mathrm{n}}\right) \bullet \left(1 + IG_{Ø}\right) \bullet \left(1 + TG_{Ø}\right) - 1$$

where

TA_g	=	Transaction effect attributable to group g, based on arithmetic method
$AA_{Ø}$	=	Active return, calculated based on arithmetic method
TG_g	=	Transaction effect attributable to group g, based on geometric method
$AG_{Ø}$	=	Active return, calculated based on geometric method

Multiple Period

The methodology for transaction-based multiperiod attribution is the same as that of holding-based attribution, except that transaction-based portfolio returns are used in place of their holding-based counterparts. To calculate the transaction-based multiple period contribution, please substitute the holding-based contribution with the transaction-based contribution in the multiperiod holding-based contribution formula.

Appendix C: Residual

Overview
The residual, also known as the return gap, is the portion of the return that cannot be explained. In a holding-based attribution analysis, it is the return that cannot be explained by the holdings composition at the beginning of the analysis period and is usually caused by intraperiod portfolio transactions, security corporate actions, and so on. In order to measure the residual, returns must be available for all securities in the portfolio and benchmark, including nonequity securities such as cash-equivalent securities. Performance attribution analysis also must be performed on the total portfolio and not just the equity portion of the portfolio. The main portion of this document focuses only on the equity portion of the portfolio. This section addresses full portfolio top-down attribution analysis, as bottom-up attribution analysis is meaningful only for the equity portion of a portfolio.

Appendix C: Residual (Continued)

Top-Down Approach, Arithmetic Method

In a top-down arithmetic attribution, residuals are defined in the formulas below. A residual is the difference between the actual return and the calculated return, the latter is based on the holdings as of the beginning of the period. This is intuitive, as the actual return only differs from its counterpart if transactions or corporate actions have occurred during the holding period.

[49] $$GA^P = R^P - R^P_{Ø} + EXP^P$$

[50] $$GA^B = R^B - R^B_{Ø}$$

[51]

$$AA = R^P - R^B = AA_{Ø} + GA^P - GA^B - EXP^P = \sum_{n=1}^{M} EA_{Ø,n} + GA^P - GA^B - EXP^P$$

where

GA^P	=	The portfolio's residual, calculated based on arithmetic method
GA^B	=	The benchmark's residual, calculated based on arithmetic method
AA	=	The portfolio's active return, calculated based on arithmetic method
R^P	=	The portfolio's actual return, when performing attribution analysis on the total portfolio
$R^P_{Ø}$	=	The portfolio's calculated return, based on formula [4]
EXP^P	=	The portfolio's net prospectus expense ratio
R^B	=	The benchmark's actual return, when performing attribution analysis on the total portfolio
$R^B_{Ø}$	=	The benchmark's calculated return, based on formula [3]
$AA_{Ø}$	=	The portfolio's calculated active return, based on equity holdings, calculated based on arithmetic method
$EA_{Ø,n}$	=	Effect attributable to the total equity portfolio at decision level n, based on arithmetic method

Appendix C: Residual (Continued)

Top-Down Approach, Geometric Method

In a top-down geometric attribution, residuals are defined in the formulas below. Similar to their arithmetic counterparts, a geometric residual is the geometric difference between the actual return and the calculated return.

[52] $$GG^P = \frac{1+R^P}{1+R_{Ø}^P} \bullet \left(1+EXP^P\right) - 1$$

[53] $$GG^B = \frac{1+R^B}{1+R_{Ø}^B} - 1$$

[54] $$AG = \frac{1+R^{\mathrm{P}}}{1+R^{\mathrm{B}}} - 1 = \frac{1+AG_{Ø}}{1+EXP^P} \bullet \frac{1+GG^P}{1+GG^B} - 1 = \prod_{n=1}^{M} \frac{1+EG_{Ø,n}}{1+EXP^P} \bullet \frac{1+GG^P}{1+GG^B} - 1$$

here

GG^P	=	The portfolio's residual, calculated based on geometric method
GG^B	=	The benchmark's residual, calculated based on geometric method
AG	=	The portfolio's active return, calculated based on geometric method
R^P	=	The portfolio's actual return, when performing attribution analysis on the total portfolio
$R_{Ø}^P$	=	The portfolio's calculated return, based on formula [4]
EXP^P	=	The portfolio's net prospectus expense ratio
R^B	=	The benchmark's actual return, when performing attribution analysis on the total portfolio
$R_{Ø}^B$	=	The benchmark's calculated return, based on formula [3]
$AG_{Ø}$	=	The portfolio's calculated active return, based on equity holdings, calculated based on geometric method
$EG_{Ø,n}$	=	Effect attributable to the total equity portfolio at decision level n , based on geometric method

Appendix C: Residual (Continued)

Multiperiod Geometric Residuals

Residuals from single-period geometric attribution analysis can be linked over multiple periods to form an overall result. These formulas are not applicable to residuals calculated using the arithmetic method.

[55] $$GG_{T,Cum}^{P} = \prod_{t=1}^{T}(1+GG_{t}^{P})-1$$

[56] $$GG_{T,Ann}^{P} = (1+GG_{T,Cum}^{P})^{\frac{y}{m}}-1$$

[57] $$GG_{T,Cum}^{B} = \prod_{t=1}^{T}(1+GG_{t}^{B})-1$$

[58] $$GG_{T,Ann}^{B} = (1+GG_{T,Cum}^{B})^{\frac{y}{m}}-1$$

where

$GG_{T,Cum}^{P}$	=	Cumulative residual of the portfolio, calculated based on geometric method, cumulative from single holding periods 1 to T
$GG_{T,Ann}^{P}$	=	Annualized residual of the portfolio, calculated based on geometric method, over the time period from 1 to T
$GG_{T,Cum}^{B}$	=	Cumulative residual of the benchmark, calculated based on geometric method, cumulative from single holding periods 1 to T
$GG_{T,Ann}^{B}$	=	Annualized residual of the benchmark, calculated based on geometric method, over the time period from 1 to T
y	=	The number of periods in a year; for example, it is 12 when data are in monthly frequency
m	=	The total number of periods; for example, it is 40 when the entire time period spans over 40 months