What's Wrong with Multiplying by the Square Root of Twelve

Paul D. Kaplan, Ph.D., CFA
Director of Research
Morningstar Canada
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Summary

A commonly used measure of return volatility is the standard deviation of monthly returns multiplied by $\sqrt{12}$ to express it in the same unit as annual return. However, some researchers have noted that multiplying by $\sqrt{12}$ is not mathematically correct and present the correct method. This article reviews the motivation for multiplying by $\sqrt{12}$ and explains why it does not apply to returns. It then shows the derivation of the correct method to help the reader understand why it is in fact correct. It then illustrates the bias introduced by multiplying by $\sqrt{12}$ rather than using the correct method. Finally, it presents two alternative measures of return volatility in which multiplying by $\sqrt{12}$ is the correct way to annualize the monthly measure.

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Introduction

A commonly used measure of return volatility is the standard deviation of monthly returns multiplied by $\sqrt{12}$ to express it in the same unit as annual return. However, some researchers have noted that multiplying by $\sqrt{12}$ is not mathematically correct and present the correct method. (See Tobin 1965, Levy and Gunthorpe 1993, and Morningstar 2012). In this article, I review the motivation for multiplying by $\sqrt{12}$ and explain why it does not apply to returns. I then show the derivation of the correct method to help the reader understand why it is in fact correct. I then illustrate the bias introduced by multiplying by $\sqrt{12}$ rather than using the correct method. Finally, I present two alternative measures of return volatility in which multiplying by $\sqrt{12}$ is the correct way to annualize the monthly measure.

<u>When Multiplying by $\sqrt{12}$ Works and When It Doesn't</u>

Suppose that a random variable Y is the sum of 12 independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots, X_{12}$:

$$Y = X_1 + X_2 + \cdots + X_{12} \tag{1}$$

It is a mathematical truism that the standard deviation of Y is simply $\sqrt{12}$ times the standard deviation of X:

$$Std[Y] = \sqrt{12} \times Std[X] \tag{2}$$

So if an annual return was the <u>sum</u> of monthly returns, multiplying by $\sqrt{12}$ would be the correct method to annualize standard deviation. However, an annual return is not the sum of monthly returns; rather, it is the result of a <u>product</u>. Letting R_A denote the annual return and $R_1, R_2, \ldots, R_{12}$ denote the constituent monthly returns, the relationship is:

$$R_A = (1 + R_1)(1 + R_2) \cdots (1 + R_{12}) - 1 \tag{3}$$

Since this is not a sum, it should be self-evident that multiplying by $\sqrt{12}$ cannot be correct.

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The Correct Formula

Fortunately, there is a correct method for annualizing the standard deviation of monthly returns that can be easily derived. The first step is to note that standard deviation of any random variable X can be written as follows:

$$Std[X] = \sqrt{E[X^2] - E[X]^2} \quad [4]$$

Where $E[.]$ is the mathematical expectation operator. Since adding a constant to a random variable does not change its standard deviation, we can apply equation [4] to the annual return follows:

$$Std[R_A] = Std[1 + R_A] = \sqrt{E[(1 + R_A)^2] - E[1 + R_A]^2} \quad [5]$$

Since the expected value of i.i.d. random variables is the product of their expected values, from equation [3] it follows that:

$$E[1 + R_A] = E[1 + R_1]E[1 + R_2] \cdots E[1 + R_{12}] = (1 + M_m)^{12} \quad [6]$$

where M_m is the expected value of monthly returns. Similarly, we have

$$E[(1 + R_A)^2] = E[(1 + R_1)^2]E[(1 + R_2)^2] \cdots E[(1 + R_{12})^2] = Q^{12} \quad [7]$$

where $Q = E[(1 + R_m)^2]$ for any month m. Applying equation [4] to $1+R_m$ and solving for $E[(1 + R_m)^2]$, we have:

$$Q = S_m^2 + (1 + M_m)^2 \quad [8]$$

where S_m is the standard deviation of monthly returns.

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By putting together equations [5] – [8], we can derive the correct formula for annualizing monthly standard deviations:

$$Std[R_A] = \sqrt{(S_m^2 + (1 + M_m)^2)^{12} - (1 + M_m)^{24}} \quad [9]$$

Equation [9] was derived by Tobin (1965), used by Levy and Gunthorpe (1993), and has been the method of choice for many years at Ibbotson Associates and Morningstar. (See Morningstar 2012.) However, it has not had widespread acceptance.

Biases of Multiplying by $\sqrt{12}$

A notable feature of equation [9] is that it shows that in order to calculate the annual return standard deviation, it is not enough to know the monthly standard deviation; we need to know the monthly average return as well. Hence the bias from multiplying by $\sqrt{12}$ is a function of the average monthly return as well as the standard deviation.

To see the possible extent of the bias, consider a monthly return series with a standard deviation of 6 percent. (Many equity indexes have a long-run monthly standard deviation of around this value.) Multiplying 6 percent by $\sqrt{12}$ yields an annual standard deviation of 20.78 percent. Figure 1 plots the monthly average return versus the different between this and the correct value.

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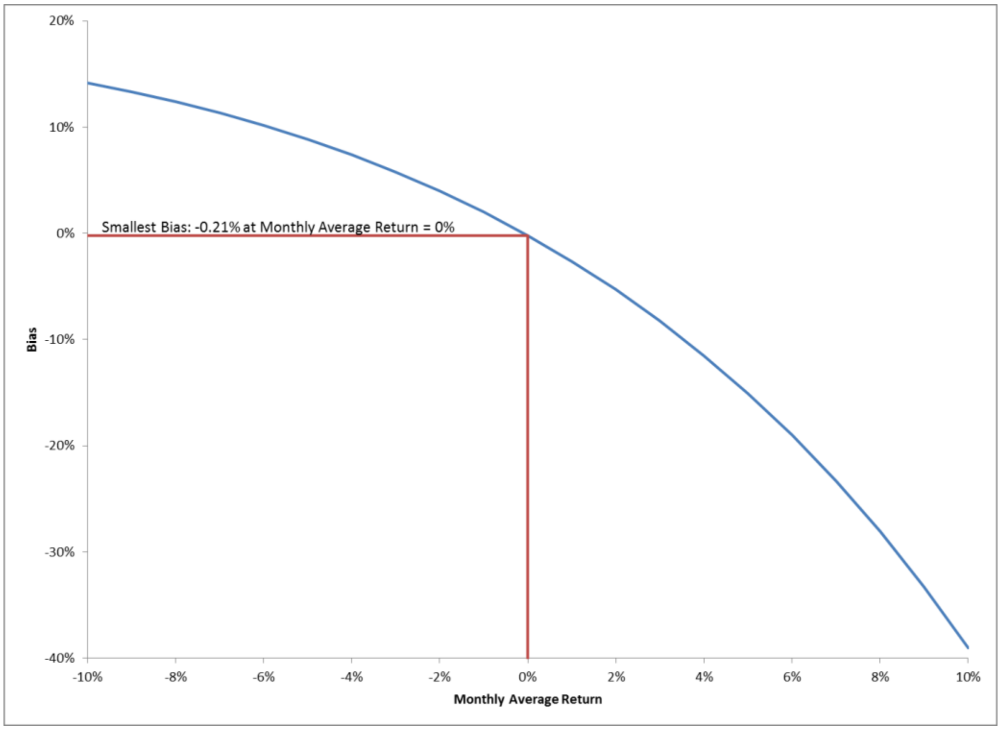
Figure 1: **Monthly Average Return vs. the Bias of Multiplying by $\sqrt{12}$**



As Figure1 shows, the bias is smallest when the average return is zero; a mere 21 basis points when the monthly standard deviation is 6 percent. However, the bias is extreme for extreme values of the average return. If the average return is extremely negative, multiplying by $\sqrt{12}$ results in an extreme overstatement of volatility. If average return is extremely positive, the bias is an extreme understatement of volatility.

These extreme biases at extreme average returns reflect the asymmetric nature of return distributions. While there is no theoretical upper limit on a return, the return on an unlevered investment cannot fall below -100 percent. Hence, if the average return is extremely negative, most returns must fall between -100 and 0 percent, limiting the amount of variation. However, if the average return is extremely positive, returns can vary over a wide range.

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Alternative Measures of Return Volatility

The biases of multiplying by $\sqrt{12}$ shown in Exhibit 1 reflect the fundamental flaw of measuring return volatility using the level of returns over a fixed time horizon such as a month. Fortunately this flaw can be easily addressed by using logarithmic returns rather than return levels. In other words, to measure volatility of R_m, use the standard deviation of $ln(1+R_m)$.

Applying the logarithmic transformation to equation [3] yields

$$ln(1+R_A) = ln(1+R_1) + ln(1+R_2) + \cdots + ln(1+R_{12}) \quad [10]$$

Notice that we now have the sum of 12 i.i.d. random variables so we can multiply by $\sqrt{12}$! Letting σ_m denote the standard deviation of $ln(1+R_m)$ and σ_A denote the standard deviation of $ln(1+R_A)$, we have

$$\sigma_A = \sqrt{12} \times \sigma_m \quad [11]$$

Note that σ_A is the measure of volatility used in the Black-Scholes formula for pricing options.

If the underlying return data is not available, it is possible to estimate σ_m from M_m and S_m and then multiple by $\sqrt{12}$ to annualize. This is because if returns follow a lognormal distribution, the standard deviation of $ln(1+R_m)$ is given by

$$\hat{\sigma}_m = \sqrt{ln(S_m^2 + (1+M_m)^2) - ln((1+M_m)^2)} \quad [12]$$

With some algebra we can show that multiplying by $\sqrt{12}$ works for this so that:

$$\hat{\sigma}_A = \sqrt{12} \times \hat{\sigma}_m \quad [13]$$

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Figure 2 plots monthly average return (M_m) versus annualized estimated logarithmic standard deviation ($\hat{\sigma}_A$), with monthly standard deviation (S_m) set at 6 percent. This figure shows that this a fairly stable measure of volatility over a wide range of average returns.

Figure 2: Monthly Average Return vs. Annualized Estimated Log-Std

To see how close σ_A and $\hat{\sigma}_A$ are in practice, I estimated them for the oldest share classes of all Canadian open-end funds for which there are returns over the 60-month period November 2007 – October 2012 using the sample mean and sample standard deviation.[1] This gave me a sample of 1,824 funds. Figure 3 plots σ_A and $\hat{\sigma}_A$ along with a pair of 45 degree lines to show for which funds the difference between σ_A and $\hat{\sigma}_A$ came to within ± 1 percent. This condition held for 1,751 or 96 percent of the funds. For the other 73 funds, there were returns that fell too far outside of the predications of the lognormal distribution to keep the difference small.

[1] I obtained the data from Morningstar Direct[SM].

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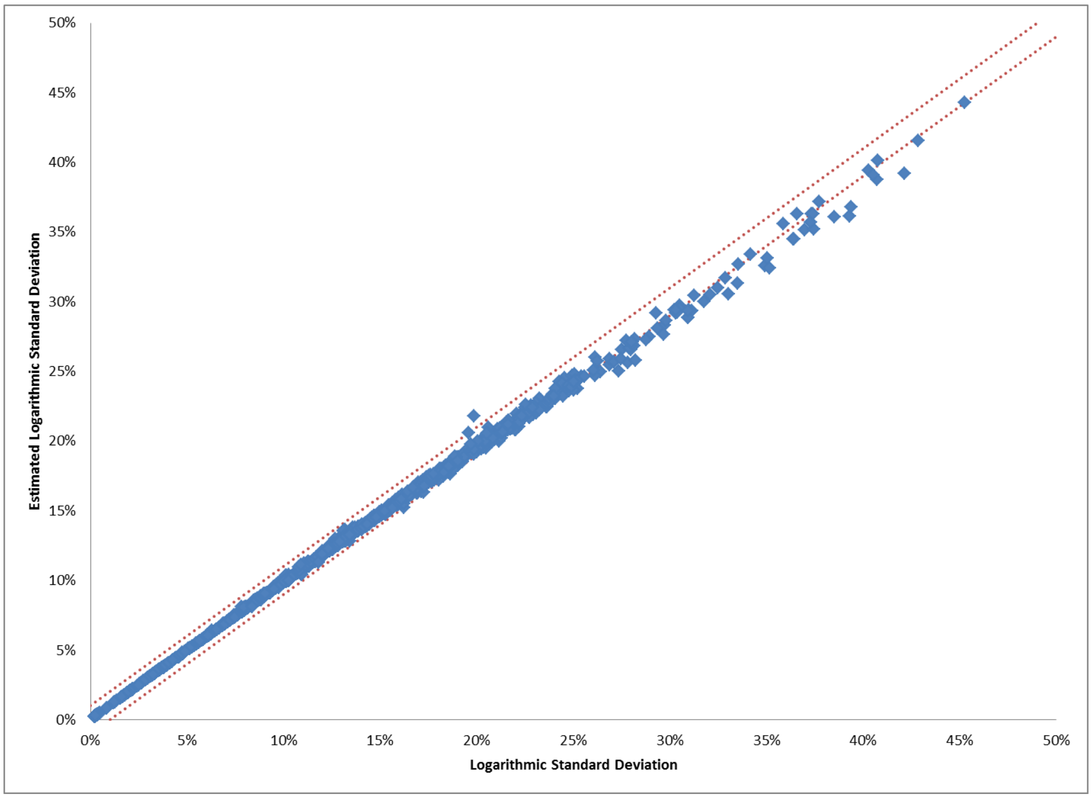
Figure 3: Direct vs. Estimated Log-Std from 60 Months of Returns on Canadian Funds



Source: Morningstar Direct[SM]

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Summary

In spite of being mathematically invalid, the most common method for annualizing the standard deviation of monthly returns is to multiply it by $\sqrt{12}$. I believe that this practice is rooted in the notion that return distributions are symmetric. However, return distributions are intrinsically asymmetric.

Fortunately there is a correct formula for annualizing monthly standard deviations. However, to use this formula, we need to know the average monthly return as well as the standard deviation. The result can be quite sensitive to the average month return due to the intrinsic asymmetric nature of return distributions.

Given that annual return relatives (i.e. returns plus 1) are products of monthly return relatives, it may be more appropriate to measure the volatility of logarithmic returns rather than level returns as is standard practice in options pricing. Since an annual logarithmic return is the sum of its monthly constituents, multiplying by $\sqrt{12}$ works.

An alternative to measuring the standard deviation of logarithmic returns directly is to estimate it indirectly from the level mean and level standard deviation using the assumption that returns follow a lognormal distribution. Multiplying by $\sqrt{12}$ is the correct method for annualizing this statistic as well. To see how well this estimation technique works, I calculated for a large and varied sample of Canadian mutual funds. For 96 percent of the funds, the estimate was within ± 1 percent of the value obtained using the direct method.

I hope that this article will encourage the abandonment of a procedure that while expedient, is clearly incorrect and to replace it with procedures that at least have the of virtue of being mathematically sound.

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